$$\Rightarrow \int \sin^2 x \; \cos x \; dx + \int \frac{-\sin y}{-\cos y} \, dy = \int 0$$

Then the general solution is:

$-\frac{\sin^3 x}{3} - \ln\,|\cos y| = c$, (Where c is an arb. cons.)

---

**<u>Ex 6 :</u> Solve** $\frac{dy}{dx} + e^x y = e^x y^2$

**<u>Solution:</u>** $\frac{dy}{dx} = e^x\, y^2 - e^x\, y \Rightarrow \frac{dy}{dx} = e^x\,(y^2 - y\,)$

$$\Rightarrow \frac{dy}{y(y-1\,)} = e^x\, dx$$

$\int \frac{1}{y(y-1\,)}\, dy$ ( We use the fraction law) ( قانون التجزئة )

$$\frac{A}{y} + \frac{B}{y-1} = \frac{A\,(y-1) + \;B\,y}{y(y-1)}$$

$\Rightarrow Ay - A + B\,y \;= 1$

$\Rightarrow (A + B)y \;= 0 \qquad \ldots(1)$

$\Rightarrow -A \;= 1 \qquad \ldots(2)$

$\Rightarrow A + B \;= 0$

$A = -B$

$From\ (2):\ A = -1$

$From\ (1): A + B = 0 \longrightarrow -1 + B = 0 \rightarrow -1 = -B \longrightarrow B = 1$

Now, $\int \frac{1}{(y^2 - y)}\, dy = \int (\frac{-1}{y} + \frac{1}{y-1})\, dy$

$= \int \frac{-1}{y} dy + \int \frac{1}{y-1}\, dy = -ln|y| + ln|y-1|$

Then the general solution is:

$-\ln|y| + \ln|y-1| = e^x + c,$ (Where c is an arb. con.)

---

## 2.2 Homogenous differential equation

**Definition2:** The function $f(x, y)$ is homogeneous function of degree $n$ if

$$f(t\, x, t\, y) = t^n\, f(\, x, y) \qquad \ldots \ldots .. \qquad \text{(I)}$$

where $t$ is a constant.

---

**Definition3:** The differential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad \text{....} \qquad \text{(II)}$$

is called homogenous if $M(x,y)$ and $N(x,y)$ are homogeneous functions of the same degree.

---

To solve the homogeneous differential equation:

$$M(x,y)dx + N(x,y)dy = 0$$

Can be written in the form

$$\frac{dy}{dx} = f(\frac{y}{x}) \qquad \text{....} \qquad \text{(III)}$$

Let $v = \frac{y}{x}$ , then equation (III) becomes

$$\frac{dy}{dx} = f(v) \qquad \text{....} \qquad \text{(IV)}$$

Since $v = \frac{y}{x}$ , then $y = vx$, and so

$$dy = vdx + xdv$$

We substitute the above expression.

The new equation is separable, we solve it and in the last we put $v = \frac{y}{x}$ to get the general solution.

---

**<u>Ex 1 :</u> Solve** $\boldsymbol{x\,y^2\,dy - (x^3 + y^3)dx = 0}$

**<u>Solution:</u>** $M(x,y) = (x^3 + y^3)$

$N(x,y) = xy^2$

$M(t\,x, t\,y) = (tx)^3 + (ty)^3 = t^3x^3 + t^3y^3$

$= t^3(x^3 + y^3) = t^3M(x,y)$, الدالة متجانسة من الدرجة الثالثة

$N(t\,x, t\,y) = t\,x\,(t\,y)^2 = tx\,(t^2\,y^2)$

$= t^3(x\,y^2) = \ t^3\,N(x,y)$, الدالة متجانسة من الدرجة الثالثة

So, $M$ and $N$ are both homogeneous, and have the same degree, so the diff. eq. is homogeneous.

Let $y = vx \longrightarrow dy = v\,dx + x\,dv$

بالتعويض في المعادلة التفاضلية نحصل على:

$xv^2x^2(vdx + xdv) - (x^3 + v^3x^3)dx = 0$

$$\Rightarrow x^3v^3dx + x^4v^2dv - x^3dx - v^3x^3dx = 0$$

$$\Rightarrow [x^4v^2dv - x^3dx = 0] * \frac{1}{x^4}$$

$$\Rightarrow \int v^2dv - \int \frac{1}{x}dx = \int 0$$

$$\Rightarrow \frac{v^3}{3} - \ln|x| = c$$

When $v = \frac{y}{x} \Rightarrow \frac{y^3}{3x^3} - \ln|x| = c$, (The general solution)

---

**<u>Ex 2 :</u> Solve** $\boldsymbol{xy\,dx + (x^2 - 2y^2)dy = 0}$

**<u>Solution:</u>** The equation is homogeneous (prove it), so

Let $v = \frac{y}{x} \longrightarrow y = v\,x \longrightarrow dy = v\,dx + x\,dv$

$$x(vx)\,dx + (x^2 - 2v^2x^2)(v\,dx + x\,dv) = 0$$

$$\Rightarrow vx^2\,dx + x^2v\,dx + x^3dv - 2v^3x^2dx - 2v^2x^3dv = 0$$

$$\Rightarrow 2vx^2\,dx - 2v^3x^2dx + x^3(1 - 2v^2)dv = 0$$

$$\Rightarrow 2x^2(v - v^3)dx + x^3(1 - 2v^2)dv = 0$$

$$\Rightarrow [2x^2(v-v^3)dx + x^3(1-2v^2)dv = 0\,] * \frac{1}{x^3(v-v^3)}$$

$$\Rightarrow \frac{2x^2}{x^3}dx + \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\Rightarrow \int \frac{2}{x}dx + \int \frac{1-2v^2}{v-v^3}\,dv = 0$$

$$\int \frac{1-2v^2}{v-v^3}\,dv \ = ?$$

$$\frac{1-2v^2}{v-v^3} = \frac{1-2v^2}{v(1-v^2)}$$

$$= \frac{1-2v^2}{v(1-v)(1+v)}$$

$$= \frac{A}{v} + \frac{B}{1-v} + \frac{C}{1+v} \qquad \ldots (*)$$

$$= \frac{A(1-v^2) + Bv(1+v) + cv(1-v)}{v(1-v)(1+v)}$$

$$= \frac{A - Av^2 + Bv + Bv^2 + Cv - Cv^2}{v(1-v^2)}$$

$$-A + B - C = -2 \qquad \ldots \ (1)$$

$$B + C \ = \ 0 \qquad \ldots \ (2)$$

$A = 1 \qquad \ldots \quad (3)$

By substituting (3) in (1), we get:

$-1 + B - C = -2 \longrightarrow B - C = -1 \qquad \ldots. \quad (4)$

Eq. (2) + Eq. (4):

$$B - C = -1$$

$$B + C = 0$$

$2B = -1 \longrightarrow B = \dfrac{-1}{2}$

From Eq. (2) $\longrightarrow C = \frac{1}{2}$

So, $A = 1,\ B = \frac{-1}{2}, and\ C = \frac{1}{2}$

Substituting A,B,C in eq.(*), we get:

بتعويض قيم $A, B, C$ في المعادلة (*)، نحصل على:

$$\int \frac{1 - 2v^2}{v - v^3}\, dv = (\frac{1}{v} + \frac{\frac{-1}{2}}{1 - v} + \frac{\frac{1}{2}}{1 + v})dv$$

$$= \ln v - \frac{1}{2}\ln|1 - v| + \frac{1}{2}\ln|1 + v| = c$$

$$\therefore 2\ln|x| + \ln|v| - \frac{1}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| = c$$

$$\Rightarrow 2\ln|x| + \ln\left|\frac{y}{x}\right| - \frac{1}{2}\ln\left|1-\frac{y}{x}\right| + \frac{1}{2}\ln\left|1+\frac{y}{x}\right| = c, \quad \text{(The general sol.)}$$

---

**Ex 3 : Prove the following diff. eq. is homogeneous, then find the general solution:**

$$\boldsymbol{xdy - ydx = \sqrt{x^2+y^2}\,dx}$$

**Solution:** We must prove the degree of

$$M(x,y)dx = N(x,y)dy$$

$$x\,dy = \left(y + \sqrt{x^2+y^2}\right)dx \qquad (1)$$

$$\because M(tx,ty) = (ty + \sqrt{(tx)^2+(ty)^2}\,)$$

$$= ty + \sqrt{t^2(x^2+y^2)}$$

$$= ty + t\sqrt{x^2+y^2}$$

$$= t(y + \sqrt{x^2+y^2}\,)$$

$$= tM(x,y),$$

$\Rightarrow\ M(x,y)$, متجانسة من الدرجة الاولى

$\because\ N(tx,ty) = tx$

$\Rightarrow\ N(x,y)$, متجانسة من الدرجة الاولى

Then the diff. eq. is homogeneous. Now, we find the general solution:

Let $y = vx \longrightarrow dy = vdx + x\,dv$ , by substituting in the diff. eq.(1), we get:

$$x\,(vdx + x\,dv) - vx\,dx = \sqrt{x^2 + v^2x^2}dx$$

$$xvdx + x^2dv - vxdx = x\sqrt{1 + v^2}dx$$

$$[x^2dv = x\sqrt{1 + v^2}dx] * \frac{1}{x^2\sqrt{1 + v^2}}$$

$$\int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{1}{x}\,dx$$

$$\int \frac{dv}{\sqrt{1 + v^2}} = ?$$

Let $v = \tan\theta \ \longrightarrow dv = \sec^2\theta\,d\theta$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{sec\,\theta}\,d\theta = \int \sec\,\theta\,d\theta$$

$$= \ln|sec\theta + \tan\theta| = \ln\left|v + \sqrt{1+v^2}\right| = l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right|$$

Then the general solution is:

$$\therefore \quad l\,n\left|\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}}\right| = \ln x + c \ \text{ (c is an arb. cons.)}$$

## 2.3 Differential Equation with Linear Coefficients (Equation that reduce to homogeneous equation)

These equations can be expressed as:

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad \dots \quad (V)$$

**Two lines:**

$a_1x + b_1y + c_1 = 0$, and $a_2x + b_2y + c_2 = 0$ $\qquad \dots \quad (VI)$

**a)** Intersect if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, (المستقيمان متقاطعان)

**or:** Intersect if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$

**b)** Parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, (المستقيمان متوازيان)

**or:** Parallel if $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

---

**Case (1):** If the two lines that in $(VI)$ intersect , i.e. $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, we seek a translation of axes using the form:

$x = x_1 + h \longrightarrow dx = dx_1$

$y = y_1 + k \longrightarrow dy = dy_1$

where $(h, k)$ is the point of intersection

Then the substitution $x = x_1 + h, y = y_1 + k \ \ (dx = dx_1, dy = dy_1)$

transform equation $(V)$ into the homogeneous equation.

**Remark: First we find the intersection point.**

**Ex 1 : Solve the following O.D.E. :**

$$(\mathbf{2x - 3y + 4})\boldsymbol{dx} + (\mathbf{3x - 2y + 1})\boldsymbol{dy} = \mathbf{0}$$

**Solution:** $\left.\begin{array}{l} 2x - 3y + 4 = 0 \\ 3x - 2y + 1 = 0 \end{array}\right]$ $\quad \frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-3}{-2}$

$\frac{2}{3} \neq \frac{-3}{-2}$ $\longrightarrow$ The two lines are intersection.

**<u>or</u>:** $\begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} = 5 \neq 0 \longrightarrow$ The two lines intersect.

Now, we must find the intersection point:

$$2x - 3y + 4 = 0 * (-3) \qquad (1)$$
$$3x - 2y + 1 = 0 \ * (2) \qquad (2)$$

$\longrightarrow 5y - 10 = 0 \ \longrightarrow y = 2$, [by substituting in Eq.(1)]

$\longrightarrow 2x - 3(2) + 4 = 0 \ \longrightarrow x = 1$

$\therefore (h, k) = (1,2)$ [ The intersection point]

Let $\quad y = y_1 + 2 \ \longrightarrow dy = dy_1$

$\quad x = x_1 + 1 \ \longrightarrow dx = dx_1$

By substituting in the D.E., we get:

$$(2(x_1 + 1) - 3(y_1 + 2) + 4)dx_1 + (3(x_1 + 1) - 2(y_1 + 2) + 1)dy_1 = 0$$

$$(2x_1 + 2 - 3y_1 - 6 + 4)dx_1 + (3x_1 + 3 - 2y_1 - 4 + 1)dy_1 = 0$$

$$(2x_1 - 3y_1)dx_1 + (3x_1 - 2y_1)dy_1 = 0$$

The above equation is homogeneous

Let $y_1 = vx_1 \longrightarrow dy_1 = v\,dx_1 + x_1\,dv$

$$(2x_1 - 3vx_1)dx_1 + (3x_1 - 2vx_1)(vdx_1 + x_1dv) = 0$$

$$(2x_1 - 3vx_1)dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$2x_1dx_1 - 3vx_1dx_1 + 3x_1vdx_1 + 3x_1^2dv - 2v^2x_1dx_1 - 2vx_1^2dv = 0$$

$$[2x_1(1 - v^2)dx_1 + x_1^2(3 - 2v)dv = 0] * \frac{1}{x_1^2(1 - v^2)}$$

$$\frac{2}{x_1}dx_1 + \frac{3 - 2v}{1 - v^2}dv = 0$$

$$2\int \frac{dx_1}{x_1} + \int \frac{3 - 2v}{(1 + v)(1 - v)}dv = \int 0$$

$$\because \frac{3 - 2v}{(1 + v)(1 - v)} = \frac{A}{(1 + v)} + \frac{B}{(1 - v)} = \frac{A - Av + B + Bv}{(1 + v)(1 - v)}$$

$$= \frac{(A + B) + (B - A)v}{(1 + v)(1 - v)}$$

$\therefore \quad A + B = 3$

$\quad B - A = -2$

$\longrightarrow B = \frac{1}{2}$, and $A = \frac{5}{2}$

$$2\int \frac{dx_1}{x_1} + \int (\frac{\frac{5}{2}}{(1+v)} + \frac{\frac{1}{2}}{(1-v)})\, dv = \int 0$$

$$4\int \frac{dx_1}{x_1} + 5\int \frac{dv}{(1+v)} + \int \frac{dv}{(1-v)} = \int 0$$

$$4\ln|x_1| + 5\ln|1+v| - \ln|1-v| = c_1$$

$$\ln x_1^4 + \ln(1+v)^5 - \ln(1-v) = c_1$$

$$\ln x_1^4 + \ln\frac{(1+v)^5}{(1-v)} = c_1$$

$\ln(x_1^4 . \frac{(1+v)^5}{(1-v)}) = c_1$ ($c_1$ is an arb. cons.)

$x_1^4 . \frac{(1+v)^5}{1-v} = e^{c_1}$, let $e^{c_1} = c$

When $v = \frac{y_1}{x_1} \longrightarrow x_1^4 . \frac{\left(1+\frac{y_1}{x_1}\right)^5}{1-\frac{y_1}{x_1}} = c$

$$\longrightarrow x_1^4 . \frac{\frac{(x_1+y_1)^5}{x_1^5}}{\frac{x_1-y_1}{x_1}} = c$$

$$\longrightarrow (x_1+y_1)^5 = C(x_1-y_1)$$

$\longrightarrow (x-1+y-2)^5 = C(x-1-(y-2))$

$\longrightarrow (x+y-3)^5 = C(x-y+1),$ (The general solution)

---

**<u>Case (2):</u>** If the two lines that in (VI) are parallel ( i.e. $\frac{a_1}{a_2}=\frac{b_1}{b_2}$ )then the solution will be using the hypothesis $\boldsymbol{z = ax + by}$ as shown in the following example

**<u>Ex 2 :</u> Solve the following O.D.E.** $(\boldsymbol{x-y+2})\boldsymbol{dx} = (\boldsymbol{x-y-3})\boldsymbol{dy}$

نلاحظ انها معادلة تفاضلية ذات معاملات خطية.

**<u>Solution:</u>**

$\because \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = -1+1=0$

اذن المستقيمان متوازيان

$(x-y+2)dx-(x-y-3)dy=0$

Let $z = x-y \longrightarrow dz = dx-dy \longrightarrow dy = dx-dz$

$\longrightarrow (z+2)dx-(z-3)dy=0$

$zdx+2dx-(z-3)(dx-dz)=0$

$$zdx + 2dx - zdx + zdz + 3dx - 3dz = 0$$

$$\int 5dx + \int (z-3)dz = \int 0$$

$$5x + \frac{z^2}{2} - 3z = c$$

$5x + \frac{(x-y)^2}{2} - 3(x-y) = c,$ (The general solution)

**ملاحظة: يمكن حل المثال السابق( Ex 2) بأسلوب اخر.**

$$(\boldsymbol{x - y + 2})\boldsymbol{dx} = (\boldsymbol{x - y - 3})\boldsymbol{dy}$$

$\longrightarrow \frac{dy}{dx} = \frac{x-y+2}{(x-y-3)}$ … (1)

Let $z = x - y$. To solve for $\frac{dy}{dx}$, we differentiate $z = x - y$ with respect to (w.r.t.) $x$ to obtain $\frac{dz}{dx} = 1 - \frac{dy}{dx}$, and so $\frac{dy}{dx} = 1 - \frac{dz}{dx}$, substitute into Eq. (1) yields:

$$1 - \frac{dz}{dx} = \frac{z+2}{z-3} \Rightarrow \frac{dz}{dx} = 1 - \frac{z+2}{z-3} = \frac{-5}{z-3}$$

$$\Rightarrow (z-3)dz = -5dx \Rightarrow \frac{z^2}{2} - 3z = -5x + c$$

$\Rightarrow \frac{(x-y)^2}{2} - 3(x-y) = -5x + c,$ (The general solution)

**Examples:**

**1) Solve the O.D.E.** $(y^2 + y)dx - (x^2 - x)dy = 0$

**Solution:**

$$\frac{dx}{x^2 - x} - \frac{dy}{y^2 + y} = 0$$

$$\underbrace{\int \frac{dx}{x^2 - x}}_{(1)} - \underbrace{\int \frac{dy}{y^2 + y}}_{(2)} = \int 0$$

Use the method of fragmentation (partial) of the fractions. نستخدم طريقة تجزئة الكسور

$$(1) \to \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$\to \frac{1}{x(x-1)} = \frac{Ax - A + Bx}{x(x-1)} = \frac{(A+B)x - A}{x(x-1)}$$

$$\to A + B = 0 \qquad \dots \quad (i)$$

$$-A = 1 \qquad \dots \quad (ii)$$

$From$: $(ii) \to A = -1$, substitute in $(i)$

$\rightarrow\ -1 + B = 0 \rightarrow B = 1$

$(2) \rightarrow \frac{1}{y(y+1)} \rightarrow A = 1$, and $B = -1$ (بنفس الأسلوب اعلاه)

$$\because \int \frac{1}{x(x-1)} dx - \int \frac{dy}{y(y+1)} = \int 0$$

$$\rightarrow \int \frac{-1}{x} dx + \int \frac{1}{x-1} - \int \frac{1}{y} dy + \int \frac{1}{y+1} dy = \int 0$$

$\rightarrow -\ln|x| + \ln|x-1| - \ln|y| + \ln|y+1| = c,$ (The general solution)

---

**2) Find the general solution of $2x^2y' - y(2x + y) = 0$**

المعادلة متجانسة ( اثبت ذلك؟) **(H.W.)**

ويمكن حل المعادلة المتجانسة بأسلوب اخر وذلك بتحويل المعادلة بصورة $f\left(\frac{y}{x}\right)$ حيث ان $v = \frac{y}{x}$

$$2x^2y' = 2xy + y^2 \rightarrow y' = \frac{2xy + y^2}{2x^2}$$

$$\rightarrow y' = \frac{y}{x} + \frac{1}{2}\left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x} \rightarrow y = vx \rightarrow dy = vdx + xdv \rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\rightarrow v + x\frac{dv}{dx} = v + \frac{1}{2}v^2$$

$$\rightarrow \; x\frac{dv}{dx} = \frac{1}{2}v^2$$

$$\rightarrow \int \frac{dv}{\frac{1}{2}v^2} = \int \frac{dx}{x}$$

$$\rightarrow \; \frac{-2}{v} = \ln|x| + c$$

$$\rightarrow \; \frac{-2x}{y} = \ln|x| + c, \qquad \text{(The general solution)}$$

---

**(3) Solve the following diff. eq. $(x^2 + y^2)dx - 2xydy = 0$**

**at $y(2) = 0$.**

**<u>Solution:</u>**

$\because M(x, y) = x^2 + y^2$

$\rightarrow \; M(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 M(x, y)$

$\because \; N(x, y) = -2x\, y$

$$\to N(tx,ty) = -2(tx)(ty) = -2t^2x\,y = t^2N(x,y)$$

(اذن المعادلة التفاضلية متجانسة)

$$[(x^2+y^2)dx - 2xydy = 0] \div x^2$$

$$\left(1+\left(\frac{y}{x}\right)^2\right)dx - \left(2\left(\frac{y}{x}\right)\right)dy \ = 0$$

Let $v = \frac{y}{x} \to y = vx \to dy = v\,dx + x\,dv \to \frac{dy}{dx} = v\ + x\,\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \to \frac{dy}{dx} = \frac{x^2}{2xy} + \frac{y^2}{2xy}$$

$$\to \frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y}\right) + \frac{1}{2}\left(\frac{y}{x}\right)$$

$$v + x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v}\right) + \frac{1}{2}v$$

$$x\frac{dv}{dx} = \frac{1}{2v} + \frac{1}{2}v - v$$

$$\to x\frac{dv}{dx} = \frac{1}{2v} - \frac{1}{2}v$$

$$\to x\frac{dv}{dx} = \frac{1}{2}\left(\frac{1}{v} - v\right)$$

$y' - y - x + 1 = 0$ ... (25)

Note that (25) is the differential eq.

**Remark:** There is another way to find the differential equation from the general solution group by using some linear algebra rules and finding the parameters of the arbitrary constants and making it equal to zero.

**Example (9) :** Find the differential equation of Example (8) using the (determinant method)

Sol. : $y = Ae^x - x \;\; \rightarrow Ae^x - x - y = 0$ ... (26)

$y' = Ae^x - 1 \;\rightarrow Ae^x - 1 - y' = 0$ ... (27)

$$\begin{vmatrix} e^x & -x-y \\ e^x & -1-y' \end{vmatrix} = 0$$

$$e^x(-1-y') + (x+y)e^x = 0$$

$$-e^x - e^x y' + xe^x + ye^x = 0 \quad (e^x \neq 0)$$

Then $-1 - y' + x + y = 0$ ... (28)

And this is the diff. eq.

**Example (10) :** Find the diff. eq. that the general solution is $y = c_1x + c_2x^3$using the determinant method

Sol. : $y = c_1x + c_2x^3 \;\rightarrow c_1x + c_2x^3 - y = 0$ ... (29)

$y' = c_1 + 3c_2x^2 \;\rightarrow\; c_1 + 3c_2x^2 - y' = 0$ ... (30)

$$y'' = 0 + 6c_2 x \rightarrow 0 + 6c_2 x - y'' = 0 \qquad \dots (31)$$

The det. is

$$\begin{vmatrix} x & x^3 & -y \\ 1 & 3x^2 & -y' \\ 0 & 6x & -y'' \end{vmatrix} = 0$$

$$x \begin{vmatrix} 3x^2 & -y' \\ 6x & -y'' \end{vmatrix} - 1 \begin{vmatrix} x^3 & -y \\ 6x & -y'' \end{vmatrix} + 0 \begin{vmatrix} x^3 & -y \\ 3x^2 & -y' \end{vmatrix} = 0$$

$$x(-3x^2y'' + 6xy') - (-x^3y'' + 6xy) = 0$$

$$-3x^3y'' + 6x^2y' + x^3y'' - 6xy = 0$$

$$[-2x^3y'' + 6x^2y' - 6xy = 0] \div \frac{1}{6}x$$

$$\frac{-x^2y''}{3} + xy' - y = 0$$

$$y = \frac{-x^2y''}{3} + xy' \qquad \dots (32)$$

And this is the same result in Ex.7 eq.(22)

## Exercises:

1- Find the differential equation of the following curves where A,B and C are arbitrary constants.

$a)\ y = Ax^2 + A^2$

$b)\ y = Ax^2 + Bx + C$

$c)\ y = A\sin x + B\cos x$

$d)\ y = Ae^x\cos(3x + B)$

2- Find the differential equation in which the general solution is the set of equations of the circles whose centers are located on the line $y = x$ and the radius of each is equal to 1.

3- Find the differential equation of the hyperbola $xy = c$ ; $c$ is an arbitrary constant.

4- Find the differential equation for the set of all straight lines in the plane.

5- Find the differential equation of the following parabolas $y^2 = 4p(x - h)$.

6- Find the differential equation for the set of all circles that contact with y-axis in the origin point.

## 1.6 : Existence and Uniqueness of the Solution of the differential equation.

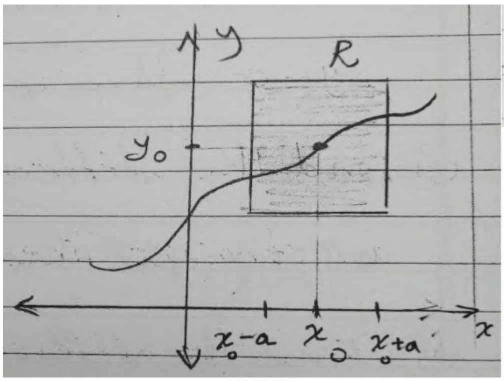
Consider the ordinary differential equation $\frac{dy}{dx} = f(x,y)$ with the initial value $y(x_0) = y_0$ where f is defined on the region

$$R = \{(x,y): |x - x_0| < a, |y - y_0| < b, \qquad a\&b > 0\}$$

If $f$ and $\frac{\partial f}{\partial y}$ are continuous on $R$ then the equation $\frac{\partial f}{\partial y} = f(x,y)$ has unique continuous solution $y = \Phi(\mathrm{x})$ passes from the point $(x_0, y_0)$ for all $x, y\ in\ R$

(i.e.)

$$\begin{cases} 1)\ If\ f\ is\ continuous\ near \\ (x_0, y_0) then\ the\ solution\ is \\ exist \\ 2)\ If\ \dfrac{\partial f}{\partial y}\ is\ continuous\ near \\ (x_0, y_0)\ then\ the\ solution\ \ is\ unique \end{cases}$$




**Example (11) :** Is the solution of the equation

$\frac{dy}{dx} = 2x \quad , y(1) = 3$ ... (33)

Exist and unique at (1,3)?

Sol. :

1) $\frac{dy}{dx} = 2x$ then $f(x,y) = 2x$

Its clear that f is continuous at all $x\&y$ in xy plane then the solution is exist at (1,3)

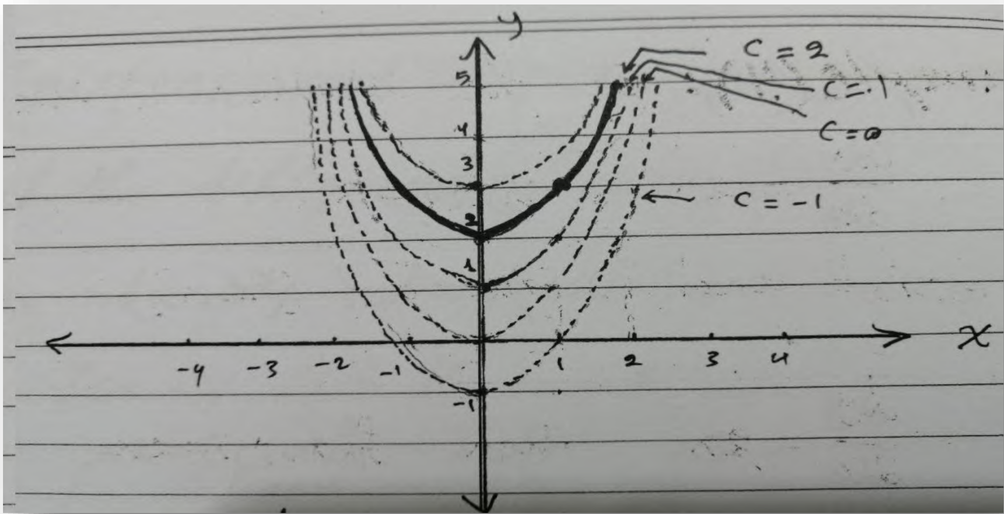
2) $\frac{\partial f}{\partial y} = 0$ and its continuous too at all $x\&y$ in xy plane then the solution is unique at (1,3) Integrating (33) to find the general Sol.

$$\frac{dy}{dx} = 2x \rightarrow dy = 2xdx \rightarrow y = x^2 + c \qquad \dots (34)$$

and this is the general solution where c is an arb. Cons.

Sub. y(1)=3 in (34), we get y(1)=$1^2 + c \rightarrow 3 = 1 + c \rightarrow c = 2$

$\therefore y = x^2 + 2$ is the solution passes from (1,3) and its clear that its unique.




Its clear that solution $y = x^2 + 2$ passes from the point (1,3) and it's the only one.

**Example (12) :** consider $x\frac{dy}{dx} = y$ discuss the existence and uniqueness of solutions

Sol. :

$$x\frac{dy}{dx} = y \rightarrow \frac{dy}{dx} = \frac{y}{x} \qquad \dots (35)$$

$$\therefore f(x, y) = \frac{y}{x} \qquad \dots (36)$$

1) Its clear that $\frac{y}{x}$ is continuous at any point $(a, b)$ where $a \neq 0$

So the solution is exist when $a \neq 0$

2) $\frac{\partial f}{\partial y} = \frac{1}{x}$ and its also continuous at any point $(a, b)$ where $a \neq 0$

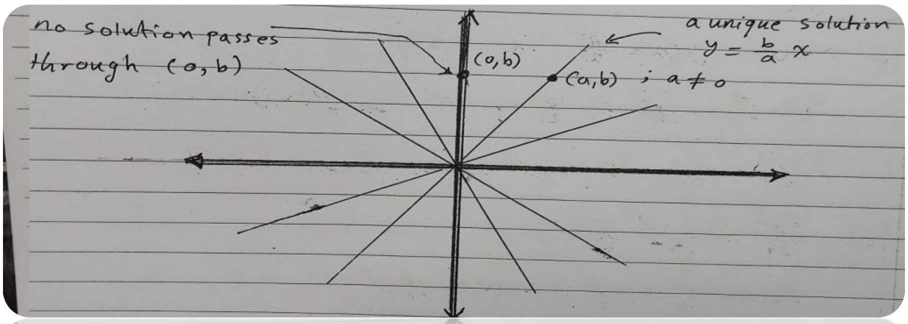
So the solution is unique when $a \neq 0$

Integrating eq. (35)

$$\frac{dy}{dx} = \frac{y}{x} \rightarrow \frac{dy}{y} = \frac{dx}{x} \rightarrow \ln y = \ln x + \ln c$$

$$\rightarrow y = cx \qquad \dots (37)$$

Eq. (37) is the general solution where c is an arbitrary constant.




The figure shows that no solution has passes from the point (0,b)

**Remark:** The point that does not satisfy the condition of existence and uniqueness is called the ( singular point ) .

**Example (13) :** Does the following Initial value problem ( Ivp ) have a unique solution?

$$\frac{dy}{dx} = e^{x} \cos y \qquad ; y(0) = \frac{\pi}{2} \qquad \ldots (38)$$

Sol: $$\frac{dy}{dx} = e^{x} \cos y \rightarrow f(x,y) = e^{x} \cos y \qquad \ldots . (39)$$

(1) sub. $\left(0, \frac{\pi}{2}\right)$ in (39)

$$f\left(0, \frac{\pi}{2}\right) = e^{0} \cos \frac{\pi}{2} = 0 \rightarrow f(x,y) = 0$$

$f(x,y)$ is continuous near $\left(0, \frac{\pi}{2}\right)$

(2) $\frac{\partial f}{\partial y} = e^{x} \sin y$

sub. $\left(0, \frac{\pi}{2}\right)$, we get

$\frac{\partial f}{\partial y} = -1$ and its continuous near $\left(0, \frac{\pi}{2}\right)$. Then there is a unique solution at $\left(0, \frac{\pi}{2}\right)$

$$\frac{dy}{dx} = e^x \cos y \rightarrow \frac{dy}{\cos y} = e^x dx \rightarrow \sec y\, dy = e^x dx$$

$$\ln|\sec y + \tan y| = e^x + c \qquad \ldots (40)$$

But $\ln|\sec y + \tan y| = \ln\left|\frac{1}{\cos y} + \tan y\right|$ $\quad and \ \cos\frac{\pi}{2} = 0$

$$\rightarrow \frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0}$$

$\therefore$ there is no solution passes from $\left(0, \frac{\pi}{2}\right)$ and exist in the general solution group. (40) we must look for a solution in another way.

$$\frac{dy}{dx} = 0 \qquad at \ \left(0, \frac{\pi}{2}\right)$$

Integrating both sides

$$y(x) = c \qquad \ldots (41)$$

Sub. $\left(0, \frac{\pi}{2}\right)$ $\rightarrow y(0) = c \rightarrow \frac{\pi}{2} = c$ ,sub. in (41) we get

$y(x) = \frac{\pi}{2}$ and this is the unique solution passes through the point $\left(0, \frac{\pi}{2}\right)$

**Example (14) :** Does the following (IVP) $\frac{dy}{dx} = x\sqrt{y-3}$

$, y(-2) = 28$ ... (42)

Have a unique solution or not?

Sol. $f(x, y) = x\sqrt{y-3}$

It's clear that f is continuous near (-2,28)

$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y-3}}$ and it's continuous near (-2,28) also.

Then the diff. eq. has a unique solution :

$\frac{dy}{dx} = x\sqrt{y-3} \rightarrow \frac{dy}{\sqrt{y-3}} = \text{xdx}$

$\rightarrow 2\sqrt{y-3} = \frac{x^2}{2} + \text{c}$ ... (43)

Sub. (-2,28) in (43), we get

$2\sqrt{28-3} = \frac{4}{2} + \text{c} \rightarrow 10 = 2 + \text{c} \rightarrow \text{c} = 8 ,$

$2\sqrt{y-3} = \frac{x^2}{2} + 8 \rightarrow \text{y} = \left(\frac{x^2}{4} + 4\right)^2 + 3$ ... (44)

Eq. (44) is the unique solution that passes through (-2,28).

# المرحلة الثانية

# المعادلات التفاضلية الاعتيادية

# Ordinary Differential Equations

## Chapter 2

# CHAPTER TWO

## The Ordinary Differential Equation of the first order and first degree

An ordinary differential equation of first order and first degree is written in one of the following forms:

المعادلة التفاضلية الاعتيادية من الرتبة الأولى والدرجة الأولى تكتب بأحدى الاشكال التالية:

$$M(x,y)dx + N(x,y)dy = 0$$

**or** $$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

**or** $$\frac{dy}{dx} = f(x,y)$$

Such that $f, M, N$ does not contain derivative.

---

Although this kind of differential equations seems simple, there is no general way of solving, but several methods depending on the type of the equation. Therefore, the equations that can be solved directly divided into several types, the most important ones are:

( رغم ان هذا النوع من المعادلات التفاضلية تبدو بسيطة الا انه لا توجد طريقة عامة للحل وانما عدة طرق حسب نوع المعادلة، وعلى ذلك تنقسم المعادلات التي يمكن إيجاد حلها بطريقة مباشرة الى عدة أنواع أهمها:)

**2.1) Separable equation.**

**2.2) Homogenous equation.**

**2.3) Differential equation with linear coefficients.**

**2.4) Exact differential equation**

**2.5) integral factors.**

**2.6) Bernoulli's equation**

**2.7) Ricatt's Equation**

**2.8) The diff. eq. of the form** $f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$

**2.9) Equation that is solved using a suitable substitution.**

Now let's start:

---

## <u>2.1 Separable of Variables:</u>

<u>**Definition1:**</u> An equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{or} \quad h(y)d\,y = g(x)\,dx$$

is said to be separable or to have separable variable. And if both $g$ and $h$ are differentiable, then:

$$\int h(y)\,dy = \int g(x)\,dx + c$$

---

**Ex 1 : Solve : a)** $\frac{dy}{dx} = 1 + e^{5x}$ **b)** $\frac{dy}{dx} = \sin(x)$

**Solution:**

**a)** $dy = (1 + e^{5x})dx \Rightarrow y = x + \frac{1}{5}e^{5x} + c$

**b)** $dy = \sin x\, dx \Rightarrow y = -\cos x + c$

---

**Ex 2 : Solve :** $(1 + x^2)\, y\, dy + (y + 3)\, dx = 0$

**Solution:** $[(1 + x^2)\, y\, dy + (y + 3)\, dx = 0] * \frac{1}{(1+x^2)(y+3)}$

$$\Rightarrow \left(\frac{y}{y+3}\right) dy + \frac{dx}{1 + x^2} = 0$$

$$\Rightarrow \int \left(1-\frac{3}{y+3}\right) dy + \tan^{-1} x = c$$

$$\Rightarrow \int dy - \int \frac{3}{y+3} dy + \tan^{-1} x = c$$

$$\Rightarrow \mathrm{y} - 3 \ln \ |\mathrm{y}+3| \ + \tan^{-1} x = c, \ c \in \mathbb{R} \qquad (c \ is \ an \ arb. cons.)$$

---

**Remark:** $\frac{y}{y+3}$ can be solved by two methods

يمكن تبسيط المقدار $\frac{y}{y+3}$ بطريقتين: اما *القسمة الطويلة* او *بالتجزئة*.

**1) القسمة الطويلة:**

$$\begin{array}{r|l} & 1 \\ \hline y+3 & y \end{array}$$

**2) التجزئة:**

$$\frac{y}{y+3} = \frac{y+3-3}{y+3} = \frac{y+3}{y+3} - \frac{3}{y+3} = 1 - \frac{3}{y+3}$$

**<u>Ex 3 :</u> Solve** $\quad \boldsymbol{y' = e^{x-y}}$

**<u>Solution:</u>** $\frac{dy}{dx} = e^x . e^{-y} \ \Rightarrow dy = e^x \ . e^{-y} \ dx$

$$\Rightarrow e^y \ dy = e^x \ dx \ \Rightarrow \ \int e^y dy = \int e^x \ dx$$

$\Rightarrow\ e^y = e^x + c$, (The general solution),(c is an arb. cons.)

---

**<u>Ex 4 :</u> Solve** $\boldsymbol{x\,y\,dy + (2yx^2 + 4x^2 - y - 2)dx = 0}$

**<u>Solution:</u>** بواسطة التحليل نحصل على:

$$[x\,y\,dy + (y+2)(2x^2-1)dx = 0] * \frac{1}{x(y+2)}$$

$$\Rightarrow \frac{y}{y+2}\,dy + \frac{2x^2-1}{x}\,dx = 0$$

$$\Rightarrow \int (1 - \frac{2}{y+2})\,dy + \int (2x - \frac{1}{x})\,dx = \int 0$$

$\Rightarrow y - 2\ln|y+2| + x^2 - \ln|x| = c$, (The general solution of O.D.E.)

---

**<u>Ex 5 :</u> Solve** $\boldsymbol{sin^2x\,cos\,y\,dx + sin\,y\,sec\,x\,dy = 0}$

**<u>Solution:</u>** $\frac{\sin^2 x}{\sec x}\,dx + \frac{\sin y}{\cos y}\,dy = 0$

$$\Rightarrow \frac{\sin^2 x}{\frac{1}{\cos x}}\,dx + \frac{-\sin y}{-\cos y}\,dy = 0$$

## *Chapter one*

**SOME IMPORTANT BASICS OF DIFFERENTIAL EQUATIONS**

**1.1:Introduction:** A differential equation is a mathematical equation that relates some function with its derivatives.

The derivatives represent their rates of change, and the equation defines a relationship between two variables.

The differential equations play an important role in many fields such as engineering, physics, economics and biology. Now, Let $x$ be a number in the domain of the function $f$ then we can express the first derivative of the function $f$ for $x$ as follows:

$$If\ \ y = f(x)\ then\ \frac{dy}{dx} = \frac{df(x)}{dx}\ or\ y' = f'(x)$$

Where the symbols $\frac{d(\dots)}{dx}$ $and$ $(\dots)'$ represent the first derivative of the function.

**1.2: Definitions**

**1.2.1: Differential equation**

A differential equation is an equation involving derivatives or differentials.

**For example:-**

$$1 - \left(\frac{dy}{dx}\right)^4 + y = x$$

$$2 - x^2\left(\frac{d^2y}{dx^2}\right)^3 + x\frac{dy}{dx} + y = 0$$

$$3-\frac{d^3y}{dx^3}-\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}=x^2+1$$

$$4-y'''+2(y'')^2+y'=cosx$$

$$5-\frac{\partial z}{\partial x}+\frac{\partial^2 z}{\partial y^2}=x^2+y$$

$$6-x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u$$

### 1.2.2: Ordinary Differential Equation

Ordinary differential equation is a differential equation involving only ordinary derivatives (i.e.) It has derivatives of one or more dependent variables w.r.t. single independent variable. Such as equations 1,2,3 and4

### 1.2.3: Partial Differential Equation

A Partial differential equation is a differential equation involving partial derivatives (i.e.) It has derivatives of one or more dependent variable w.r.t. more than one independent variable.

For example the equations 5 and 6 are p.d.e

### 1.2.4: Order of a Differential Equation

The order of the highest order derivative in a differential equation is called the order of a diff. eq.

**For example :-**

(i) Equations (1) and (6) are of order one

(ii) Equations (2) and (5) are of order two

(iii) Equations (3) and (4) are of order three

### 1.2.5: Degree of Differential Equation

The degree of differential equation that is algebraic in its derivatives is the algebraic degree of the highest derivative shown in the equation (i.e.) when the equation is free from radicals and fractions in the dependent variable and its derivatives.

**For example :-**

(i) Equations (3),(4),(5) and (6) are of first degree

(ii) Equation (2) is of the third degree

(iii) Equation (1) is of the fourth degree

Other examples:- Find the order and degree of the following differential equations:

$$1-\sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2}=\sqrt{1+\left(\frac{dy}{dx}\right)}$$

$$2-\sin(y')=y'+x+3$$

Solution (1): This equation can be written as

$$\left(\frac{d^2y}{dx^2}\right)^{2/3} = \left(1 + \frac{dy}{dx}\right)^{1/2}$$

$$\left[\left(\frac{d^2y}{dx^2}\right)^{2/3}\right]^6 = \left[\left(1 + \frac{dy}{dx}\right)^{1/2}\right]^6$$

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left(1 + \frac{dy}{dx}\right)^3$$

Therefore, this equation is of second order and fourth degree.

Solution (2): It hasn't degree since it is not algebraic in its derivatives.

### 1.2.6: Linear Differential Equation

The differential of any order shall be linear if the dependent variable and all derivatives are of the first degree and are not multiplied by each other and its general formula is

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = f(x) \qquad \ldots (1)$$

Where $a_0, a_1, \ldots, a_n$ *and* $f(x)$ are functions for $x$ on the interval $a \leq x \leq b$

An equation that is not linear is said to be nonlinear

**For example:-**

$1 - 3y^{(3)} + 2y' = 5\sin x$ $\qquad$ *Linear*

$2 - x\dfrac{d^2y}{dx^2} - y^2 = 0$ $\quad$ *non − Linear*

$3 - y^{-1}\dfrac{d^2y}{dx^2} + 8y = e^x$ $\quad$ *non − Linear*

$4 - x^2y'' + 2xy' + y = 0$ $\quad$ *Linear*

$5 - y^{(5)} + yy' + 2x = 0$ $\quad$ *non − Linear*

$6 - y'' + 5xy' + \dfrac{1}{y} = \sqrt{x+1}$ $\quad$ *non − Linear*

### 1.2.7: Homogeneous Linear Differential Equation

Equation (1) is said to be homogeneous if f(x)=0

(i.e.) $a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0$ ... (2)

Therefore, the equations (2) and (4) are homogeneous and (1), (3), (5) and (6) are non-homogeneous

**Note:** If $a_0, a_1, \dots, a_n$ in equation (1) are constant then the equation is said to be linear differential equation with constant coefficients.

**Exercises:**

Find the order, degree, linear and homogeneous of the following differential equations:

$1 - y'' + 3y' - 2y = 0$

$2 - (y''')^3 + (y'')^2 + xy = x$

$3-(y')^4+y^2=0$

$4-\sqrt[3]{(y''')^2}=\sqrt{1+(y')^2}$

$5-\dfrac{d^2y}{dx^2}-3\dfrac{dy}{dx}+4y=e^x$

## 1.3: Solution of the Differential Equation

The Solution of the differential equation is a relation between the variables of the equation and satisfies the following:

(i) Its empty of derivatives

(ii) Satisfies the differential equation

(iii) Defined on a certain interval

**Example (1):** Is $y(x)=A\sin 2x+B\cos 2x$ a solution of the diff. eq. $y''+4y=0$ ... (3)

Sol. First, we must derive the function that given twice

$$y=A\sin 2x+B\cos 2x \quad \text{... (4)}$$

$$y'=2A\cos 2x-2B\sin 2x \quad \text{... (5)}$$

$$y''=-4A\sin 2x-4B\cos 2x \quad \text{... (6)}$$

Substituting (4), (6) in (3), we get

$$-4A\sin 2x-4B\cos 2x+4(A\sin 2x+B\cos 2x)$$

$$= -4A\sin 2x - 4B\cos 2x + 4A\sin 2x + 4B\cos 2x$$

$$= 0$$

Thus, the given function satisfies the eq. (3)

$\therefore\ y(x) = A\sin 2x + B\cos 2x$ is a solution of (3)

**Example (2):** Prove that the function $y(x) = x\ln x - x$ ... (7)

is a solution of $xy' = x + y$ ... (8)

Sol. Deriving (7) w.r.t. x we get

$$y'(x) = x\frac{1}{x} + \ln x - 1$$

$$y'(x) = \ln x \qquad \text{... (9)}$$

Substituting (7),(9) in (8), we get

$$x\ln x = x + x\ln x - x$$

$$x\ln x = x\ln x$$

Hence, the equation (7) is a solution of the diff. eq. (8).

**1.3.1: General solution of the differential equation**

The general solution of the differential equation is the solution that is free of derivatives and contains a number of arbitrary constants and their number is equal to the order of the equation .

**Example (3):** Find the general solution of the equation $y''' = 0$

Sol. Integrating both sides three times

$$\int y''' \, dx = 0 \cdot dx \qquad \dots (10)$$

$$y'' = c_1 \qquad \dots (11)$$

$$y' = c_1 x + c_2 \qquad \dots (12)$$

$$y = \frac{c_1}{2} x^2 + c_2 x + c_3 \qquad \dots (13)$$

Where $c_1, c_2 \; and \; c_3$ are arbitrary constants.

Note that, the number of the constants are equal to the order of the equation

### 1.3.2: The Particular Solution

It's the solution that results after substituting the values of the arbitrary constants in the general solution.

**Example (4):** write the particular solution of the equation $y''' = 0 \; when \; c_1 = 2, c_2 = 2, c_3 = 0.$

Sol. : The solution of $y''' = 0$ is $y(x) = \frac{c_1}{2} x^2 + c_2 x + c_3$ ( from Ex(3))

Sub. $c_1, c_2 \; and \; c_3$ in it

$$y(x) = \frac{2}{2} x^2 + 2x + 0$$

$$y(x) = x^2 + 2x$$

**Remark:** A general solution is a set of solutions that represent curves and are not intersected while only one of them passes through a given point of existence of these curves and at this point one real value is determined for the arbitrary constant.

**Example (5):** Find the general solution and the particular solution of the equation $y' = x$ ... (14)

That passes through the point $(1,2)$ and sketch the integral curves.

Sol. : Integrating (14) w.r.t. x we get

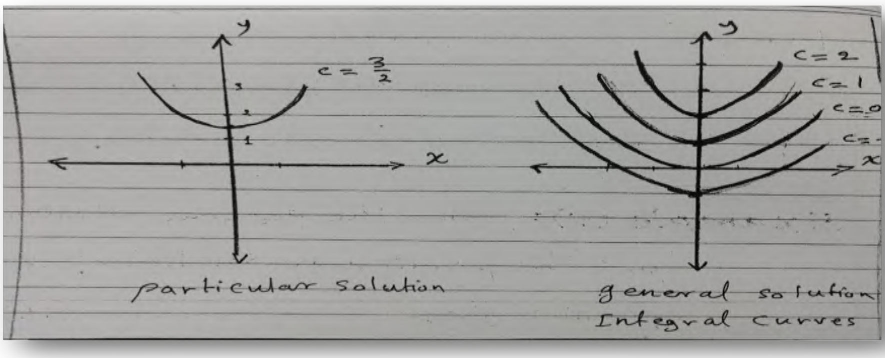
$$y = \frac{x^2}{2} + c \qquad \text{... (15)}$$

This is the general solution

To find the particular solution, substituting the point (1,2) in (15)

$$2 = \frac{1}{2} + c \;\rightarrow c = \frac{3}{2}, \text{ then } y = \frac{x^2}{2} + \frac{3}{2} \qquad \text{... (16)}$$

And this is the particular solution



## 1.4: Singular Solution of the Differential Equation

The singular solution is a solution that appears for some differential eq. and doesn't belong to the general solution group.

**Example (6):** Find the general solution and the singular solution of the equation $2y' = 3y^{1/3} \quad ; x \in R$

Sol. : $2y' = 3y^{1/3} \rightarrow 2\frac{dy}{dx} = 3y^{3}$

$\frac{dy}{y^{1/3}} = \frac{3}{2}dx \qquad ; y \neq 0$

$y^{-1/3}dy = \frac{3}{2}dx$

Integrating both sides .

$\frac{3}{2}y^{2/3} = \frac{3}{2}x + c$

$\sqrt[3]{y^2} = x + c_1 \qquad where \ \ c_1 = \frac{2c}{3}$

y= $(x + c_1)^{3/2} \qquad ; y \neq 0$ ... (17)

this is the general solution.

Now. If $y = 0$ , we note that its satisfying the diff. eq.

$y' = 0 \rightarrow 2(0) = 3(0) \rightarrow 0 = 0$

$\therefore y = 0$

is a solution to the diff. eq. but it's not belong to the general solution group, then $y = 0$ is a ***singular solution***

## Exercises :

1- Find the order, degree and linearity of the following differential equations.

$a - y' + 8xy^2 = 0$

$b - (y')^2 + xy' = y^2$

$c - \sqrt{y''} = 3y' + x$

$d - y^{(4)} = \sqrt{y'}$

$e - (y'')^{1/3} = k(1 + (y')^2)^{5/2}$

2-Prove that every equation in the list K is a solution of the differential eqs. in the list H , where A,B,C are constants.

| H | K |
|---|---|
| $1 - xy' = x^2 + y$ | $y = x^2 + cx$ |
| $2 - yy'' - (y')^2 = y^2 \ln y$ | $\ln y = Ae^x + Be^{-x}$ |
| $3 - xy' + y + x^4y^4e^x = 0$ | $y^{-3} = x^3(3e^x + c)\;; y \neq 0$ |
| $4 - y'' + 3y' + 2y = 0$ | $y = Ae^{-x} + Be^{-2x}$ |
| $5 - 2(y')^2 - xy' + y = 3$ | $\begin{cases} y = ct + 3 \\ x = 2t + c \end{cases}$ |

3-Prove that all of the equations

(i) $y = 2e^x$ $(ii) y = 3x$

$(iii) y = Ae^x + Bx \qquad , A \text{ and } B \text{ are constants.}$

are solutions of the diff. eq. $y''(1-x) + y'x - y = 0$

4- Find the value of A (if it exist) that makes $y = Ax^3$ a solution of the diff. eqs.

$a)\ x^2y'' + 6xy' + 5y = 0 \quad , b)\ x^2y'' + 6xy' + 5y = x^2$

5- What is the values of the constant C that make $y = e^{Cx}$ a solution of the equation $y'' + 5y' + 6y = 0$

6- Find the general solution and the singular solution of the equation $(y')^2 = 4y$

**1.5: Composition the Differential Equation from the General Solution**

In this subject we will discuss how to find the differential equation if we know the general solution.

The method depends on the relationship of the number of arbitrary constants in the General solution group and the order of the differential equation, where the equation is derived by a number of equal times for a number of constant as shown in the following examples.

**Example (7) :**Find the diff. eq. that the general solution is $y = c_1x + c_2x^3, where\ c_1 and\ c_2$ are arbitrary constants.

Sol. : $y = c_1x + c_2x^3$ ... (18)

Driving (18) twice, to get

$$y' = c_1 + 3c_2x^2 \qquad \dots(19)$$

$$y'' = 6c_2x \rightarrow c_2 = \frac{y''}{6x} \qquad \dots(20)$$

Sub. (20) in (19), we get

$$y' = c_1 + \frac{xy''}{2} \rightarrow c_1 = y' - \frac{1}{2}xy'' \qquad \dots(21)$$

Sub. (20) and (21) in (18)

$$y = \left(y' - \frac{1}{2}xy''\right)x + \frac{y''}{6x}x^3$$

$$= xy' - \frac{1}{2}x^2y'' + \frac{1}{6}x^2y''$$

$$= xy' - \frac{1}{3}x^2y'' \qquad \dots(22)$$

And this is the required differential equation.

**Example (8) :** Find the diff. eq. that the general solution is $y = Ae^x - x$

Sol. : $y = Ae^x - x$ ... (23)

Driving (23) , we get

$$y' = Ae^x - 1 \rightarrow Ae^x = y' + 1 \qquad \dots(24)$$

Sub. (24) in (23) we get

$$y = y' + 1 - x$$