

$$\text{or: } x - 2 \leq 0 \wedge x + 2 \leq 0 \implies x \leq 2 \wedge x \leq -2 \implies (-\infty, -2]$$

$$\text{Hence, } D_f = (-\infty, -2] \cup [2, \infty) = \mathbb{R} \setminus (-2, 2)$$

To find R_f :

$$\text{Since, } y^2 \geq 0 \implies y \in \mathbb{R}^+ \implies R_f = \mathbb{R}^+$$

Example 2 : Let $g(x) = -\sqrt{2x - 1}$, find D_g and R_g ?

To find D_g :

$$2x - 1 \geq 0 \implies 2x \geq 1 \implies x \geq \frac{1}{2} \implies D_g = [\frac{1}{2}, \infty)$$

$$y = -\sqrt{2x - 1} \implies y^2 = 2x - 1 \implies 2x = y^2 + 1 \implies x = \frac{y^2 + 1}{2}$$

To find R_g :

$$y \leq 0 \implies y \in \mathbb{R}^- \implies R_g = \mathbb{R}^- = (-\infty, 0]$$

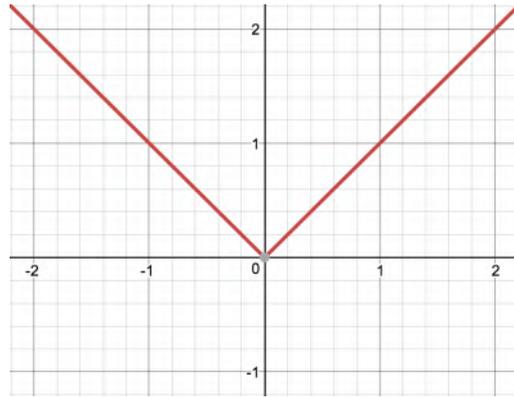
Definition: The function that is defined by more than one formula (e.g., the function are written using the brace $\{ \}$, signum function absolute value function) is) called **Piecewise function**.

Remark (3): The domain of the Piecewise function are the restrictions of the functions.

$$\text{Example 1 : Let } f(x) = |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0, \\ -x & \text{if } x < 0 \end{cases}$$

find D_f and R_f ?

$$D_f = \mathbb{R} \text{ and } R_f = \mathbb{R}^+$$



Example 2 : Let $g(x) = \begin{cases} -1 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$, find D_g and R_g ?

$$D_g = \mathbb{R}$$

$$R_g = \{-1, 3\}$$

Example 3 : Let $h(x) = y = |x + 3|$, find D_h and R_h ?

$$\text{since, } |x + 3| = \begin{cases} x + 3 & \text{if } x + 3 > 0 \rightarrow x > -3 \\ 0 & \text{if } x + 3 = 0 \rightarrow x = -3 \\ -(x + 3) & \text{if } x + 3 < 0 \rightarrow x < -3 \end{cases}$$

$$D_h = \mathbb{R}$$

$$R_h = \mathbb{R}^+$$

Example 4 : Let $f(x) = \begin{cases} x & \text{if } x < -2 \\ x + 1 & \text{if } -2 \leq x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$, find D_f and R_f ?

$$D_f = ?$$

$$x < -2 \vee -2 \leq x \leq 1 \vee x > 1$$

$$\implies (-\infty, -2) \cup [-2, 1] \cup (1, \infty) = \mathbb{R}$$

$$\implies D_f = \mathbb{R}$$

$$R_f = ?$$

$$x < -2 \vee -1 \leq x \leq 2 \vee x > 1$$

$$\implies (-\infty, -2) \cup [-1, 2] \cup (1, \infty) = \mathbb{R} \setminus [-2, -1)$$

$$\implies R_f = \mathbb{R} \setminus [-2, -1)$$

Example 5 : Let $w(t) = |t - 2|$, find D_w and R_w ?

$$|t - 2| = \begin{cases} t - 2 & \text{if } t > 2 \\ 0 & \text{if } t = 2 \\ -(t - 2) & \text{if } t < 2 \end{cases}$$

$$D_w = \mathbb{R}$$

$$R_w = \mathbb{R}^+$$

Remark (4): The domain of the Rational function is all the real number values except the value of x which makes the denominator equal to zero.

Example 1 : Let $f(x) = \frac{x}{x^2-1}$, find D_f and R_f ?

To find D_f :

$$x^2 - 1 \neq 0 \implies x^2 \neq 1 \implies \sqrt{x^2} \neq 1 \implies |x| \neq 1 \implies x \neq \mp 1$$

$$D_f = \mathbb{R} \setminus \{-1, 1\}$$

To find R_f :

$$\begin{aligned} y = f(x) = \frac{x}{x^2-1} &\implies x = yx^2 - y \implies yx^2 - x - y = 0 \\ \implies x &= \frac{1 \mp \sqrt{1+4y^2}}{2y} \quad \left(\text{Using } x = \frac{-b \mp \sqrt{b^2-4ac}}{2a}\right) \end{aligned}$$

$$\text{Since } 2y \neq 0 \implies y \neq 0,$$

$$\text{and } 1 + 4y^2 \geq 0 \implies y^2 \geq \frac{-1}{4} \implies y^2 \geq 0 \implies y \in \mathbb{R}$$

$$\text{Hence, } R_f = \mathbb{R} \setminus \{0\}$$

Example 2 : Let $h(x) = \sqrt[3]{\frac{x+1}{x-2}}$, find D_h and R_h ?

To find D_h :

$$\sqrt[3]{x-2} \neq 0 \implies x-2 \neq 0 \implies x \neq 2$$

$$\text{Hence, } D_h = \mathbb{R} \setminus \{2\}$$

To find R_h :

$$y^3 = \frac{x+1}{x-2} \implies (x-2)y^3 = x+1$$

$$\implies xy^3 - 2y^3 - x - 1 = 0$$

$$\implies (y^3 - 1)x = 2y^3 + 1$$

$$\implies x = \frac{2y^3+1}{y^3-1}$$

$$\text{Let } y^3 - 1 \neq 0 \implies y^3 \neq 1 \implies y \neq 1$$

$$\text{Hence, } R_h = \mathbb{R} \setminus \{1\}$$

Problems 2.1: Find the domains and ranges for the following functions?

1. $f(x) = \sqrt{\frac{1}{x} - 2}$

5. $f(x) = \frac{1}{x^2+1} + 3$

2. $h(x) = \frac{\sqrt{x+1}}{x-1}$

6. $w(t) = \sqrt{x^2 + 25}$

3. $l(x) = \frac{x+1}{|x-5|}$

7. $g(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$

4. $g(x) = \frac{2-x}{\sqrt{1-x}}$

Algebraic of function:-

Let f and g be two functions, then:-

1. Equality of functions:

f and g are equality $\iff D_f = D_g$ and $f(x) = g(x)$

2. The Sum of functions:

The sum of f and g is : $(f + g)(x) = f(x) + g(x)$

with the domain: $D_{f+g} = D_f \cap D_g$

3. The Difference of functions:

The difference between f and g is: $(f - g)(x) = f(x) - g(x)$

with the domain: $D_{f+g} = D_f \cap D_g$

4. The Product of functions:

The product of f and g is: $(f \cdot g)(x) = f(x) \cdot g(x)$

with the domain: $D_{f.g} = D_f \cap D_g$

5. The Division of functions:

The division of f and g is: $(f|g)(x) = \frac{f(x)}{g(x)}$

with the domain: $D_{f|g} = D_f \cap D_g \setminus \{x \in \mathbb{R} : g(x) = 0\}$

Similarly,

$$(g|f)(x) = \frac{g(x)}{f(x)}$$

$$D_{g|f} = D_g \cap D_f \setminus \{x \in \mathbb{R} : f(x) = 0\}$$

Example 1 : Which of the following functions are equal to the function

$$f(x) = \frac{x-2x^2}{x}?$$

1. $g(x) = 1 - 2x$

Solution:- $D_g = \mathbb{R}; D_f = \mathbb{R} \setminus \{0\}$

Since, $D_f \neq D_g \implies f(x) \neq g(x)$

2. $h(x) = \frac{x^2-2x^3}{x^2}$

Solution:- $D_h = \mathbb{R} \setminus \{0\}; D_f = \mathbb{R} \setminus \{0\}$

$$h(x) = \frac{x^2-2x^3}{x^2} = \frac{x(x-2x^2)}{x.x} = \frac{x-2x^2}{x} = f(x)$$

Since, $D_h = D_f$ and $h(x) = f(x) \implies h(x) = f(x)$

3. $l(x) = \sqrt{1 - 4x + 4x^2}$

Solution:-

$$\sqrt{1 - 4x + 4x^2} = \sqrt{(1 - 2x)(1 - 2x)} = \sqrt{(1 - 2x)^2}$$

$$= |1 - 2x| \implies D_l = \mathbb{R}$$

Since, $D_l \neq D_f \implies l(x) \neq f(x)$

$$4. w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)}$$

Solution:-

$x(1+x^2) \neq 0 \implies x \neq 0 \vee 1+x^2 \neq 0$ (i.g. , $x^2 \neq 0$ which is always true)

$$\implies D_w = \mathbb{R} \setminus \{0\}$$

$$w(x) = \frac{(x^3+x)(1-2x)}{x(1+x^2)} = \frac{x(x^2+1)(1-2x)}{x(1+x^2)} = \frac{x-2x^2}{x} = f(x)$$

Since, $D_w = D_f$ and $w(x) = f(x) \implies w(x) = f(x)$

Example 2 : If $f(x) = \sqrt{x+1}$ and $g(x) = \sqrt{4-x}$, find

$f(x) + g(x)$, $f(x) - g(x)$, $f(x) \cdot g(x)$, $f(x)/g(x)$, and $g(x)/f(x)$

with domain for all.

Solution:-

Since, $f(x) = \sqrt{x+1}$,

$$x+1 \geq 0 \implies x \geq -1$$

$$\implies D_f = [-1, \infty)$$

Since, $g(x) = \sqrt{4-x^2}$,

$$4-x^2 \geq 0 \implies x^2 \leq 4 \implies |x| \leq 2 \implies -2 \leq x \leq 2$$

$$\implies D_g = [-2, 2]$$

Now,

$$(f+g)(x) = f(x) + g(x) = \sqrt{x+1} + \sqrt{4-x^2}$$

$$(f - g)(x) = f(x) - g(x) = \sqrt{x+1} - \sqrt{4-x^2}$$

$$(f \cdot g)(x) = f(x) \cdot g(x) = \sqrt{x+1} \cdot \sqrt{4-x^2} = \sqrt{(x+1)(4-x^2)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{\sqrt{4-x^2}} = \sqrt{\frac{(x+1)}{(4-x^2)}}$$

$$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \frac{\sqrt{4-x^2}}{\sqrt{x+1}} = \sqrt{\frac{(4-x^2)}{(x+1)}}$$

Also,

$$D_{f+g} = D_{f-g} = D_{f \cdot g} = D_f \cap D_g = [-1, \infty) \cap [-2, 2] = [-1, 2]$$

$$D_{\frac{f}{g}} = D_f \cap D_g \setminus \{x \in \mathbb{R} : g(x) = 0\}$$

$$= [-1, 2] \setminus \{x \in \mathbb{R} : \sqrt{4-x^2} = 0\}$$

$$= [-1, 2] \setminus \{-2, 2\}$$

$$= [-1, 2)$$

$$D_{\frac{g}{f}} = D_f \cap D_g \setminus \{x \in \mathbb{R} : f(x) = 0\}$$

$$= [-1, 2] \setminus \{x \in \mathbb{R} : \sqrt{x+1} = 0\}$$

$$= [-1, 2] \setminus \{-1\}$$

$$= (-1, 2]$$

Problems 2.2:

1. Check whether each of the following two functions equal or not?

$$(a) \quad f(x) = \frac{2x^2+4x}{6x^2}, \quad g(x) = \frac{6x^3+12x^2}{6x^3}$$

$$(b) \quad v(x) = \frac{\sqrt{x+1}}{x^3}, \quad w(x) = \frac{\sqrt[3]{x^2-1}}{\sqrt{x^2}}$$

$$(c) \quad h(x) = \frac{2x^2+3x^{-2}}{8x}, \quad l(x) = \frac{2x^3+3x^{-1}}{8x^2}$$

2. Find each of $f + g$, $f - g$, $f \cdot g$, f/g , g/f , then find the domain of

each of them?

$$(a) f(x) = x^2, \quad g(x) = x + 1$$

$$(b) f(x) = x^3 + x, \quad g(x) = \frac{1}{\sqrt{x+1}}$$

$$(c) f(x) = \frac{x}{x+1}, \quad g(x) = \frac{x-1}{\sqrt{x}}$$

Composition Functions:-

Let $f(x)$ and $g(x)$ be two functions such that $R_{g(x)} \subseteq D_{f(x)}$, then there exist a function $(f \circ g)(x)$ define in the following formula:

$$(f \circ g)(x) = f(g(x))$$

$$D_{(f \circ g)(x)} = \{x : g(x) \in D_{f(x)} \wedge x \in D_{g(x)}\}$$

Similarly, we can define $(g \circ f)(x)$ as follows:

$$(g \circ f)(x) = g(f(x))$$

$$D_{(g \circ f)(x)} = \{x : f(x) \in D_{g(x)} \wedge x \in D_{f(x)}\}$$

Note: $(f \circ g)(x) \neq (g \circ f)(x)$

Example 1: Let $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, find $f \circ g$ and $g \circ f$?

Solution:-

First, we are going to find the domain and range for $f(x)$ and $g(x)$,

$$f(x) = \sqrt{x} \implies D_f = \mathbb{R}^+ = [0, \infty)$$

$$y = \sqrt{x} \implies y^2 = x \implies R_f = \mathbb{R}^+ = [0, \infty)$$

Also,

$$g(x) = x^2 + 1 \implies D_g = \mathbb{R} = [0, \infty)$$

$$y = x^2 + 1 \implies y = x^2 + 1 \implies x^2 = y - 1 \implies x = \mp\sqrt{y - 1}$$

$$\text{So, } y - 1 \geq 0 \implies y \geq 1 \implies R_g = [1, \infty)$$

To find $f \circ g$: Since $R_g = [1, \infty) \subseteq [0, \infty) = D_f$, so $f \circ g$ exist.

$$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x^2 + 1}$$

$$D_{f \circ g} = \{x : x \in D_g \text{ and } g(x) \in D_f\}$$

$$= \{x : x \in \mathbb{R} \text{ and } x^2 + 1 \in \mathbb{R}^+\}$$

$$= \{x : x \in \mathbb{R} \wedge x \in \mathbb{R}\} = \mathbb{R}$$

(Since $x^2 + 1 \geq 0 \implies x^2 \geq -1$ which is always true, and hence $x \in \mathbb{R}$)

To find $g \circ f$: Since $R_f = [0, \infty) \subseteq [0, \infty) = D_g$, so $g \circ f$ exist.

$$(g \circ f)(x) = g(f(x)) = (\sqrt{x})^2 + 1 = x + 1$$

$$D_{g \circ f} = \{x : x \in D_f \wedge f(x) \in D_g\}$$

$$= \{x : x \in \mathbb{R}^+ \wedge \sqrt{x} \in \mathbb{R}\}$$

$$= \{x : x \in \mathbb{R}^+ \wedge x \in \mathbb{R}^+\} = \mathbb{R}^+ \quad [\text{Since, } x \geq 0 \implies x \in \mathbb{R}^+]$$

Example 2: Let $f(x) = \sqrt{x-4}$ and $g(x) = \frac{x+1}{3-x}$, find $f \circ g$ and $g \circ f$?

Solution:-

First, we are going to find the domain and range for $f(x)$ and $g(x)$,

To find D_f :

$$x - 4 \geq 0 \implies x \geq 4 \implies D_f = [4, \infty)$$

To find R_f :

$$y = \sqrt{x-4} \implies y^2 = x-4 \implies x = y^2 + 4 \implies R_f = \mathbb{R}^+$$

Also,

To find D_g :

$$3 - x \neq 0 \implies x \neq 3 \implies D_g = \mathbb{R} \setminus \{3\}$$

To find R_g :

$$y = g(x) = \frac{x+1}{3-x} \implies x+1 = 3y - xy \implies x + xy = 3y - 1$$

$$\implies x = \frac{3y-1}{1+y}$$

$$\because y + 1 \neq 0 \implies y \neq -1$$

$$\text{Hence, } R_g = \mathbb{R} \setminus \{-1\}$$

To find $f \circ g$:

$$R_g = \mathbb{R} \setminus \{-1\} \not\subseteq [4, \infty) = D_f \implies f \circ g \text{ does not exist.}$$

To find $g \circ f$:

$$R_f = \mathbb{R}^+ \not\subseteq \mathbb{R} \setminus \{3\} = D_g \implies g \circ f \text{ does not exist.}$$

Problems 2.3: Find $f \circ g$ and $g \circ f$ for the following functions:-

1. $f(x) = |x|, \quad g(x) = -x$

2. $f(x) = \frac{x}{x+2}, \quad g(x) = \frac{x-1}{x}$

3. $f(t) = \sqrt{x-1}, \quad g(x) = \sqrt{1-x}$

4. $f(x) = x+1, \quad g(x) = 2x$

5. $f(x) = -\sqrt{x}, \quad g(x) = x^2 + 1$

6. $f(x) = 2x+4, \quad g(x) = \frac{1}{2}x - 2$

$$7. f(x) = x^2, \quad g(x) = 2x + 3$$

$$8. f(x) = x^3, \quad g(x) = \sqrt{1-x}$$

The Greatest Integer Function:-

The function whose value at any number x is the greatest integer less than or equal to x is called the greatest integer function. It is denoted by “ $\lceil \]$ ” such that $\lceil x \rceil \leq x$.

Examples:

$$\lceil 2 \rceil = 2$$

$$\lceil 1.5 \rceil = 1$$

$$\lceil -1.5 \rceil = -1$$

$$\lceil 3.4 \rceil = 3$$

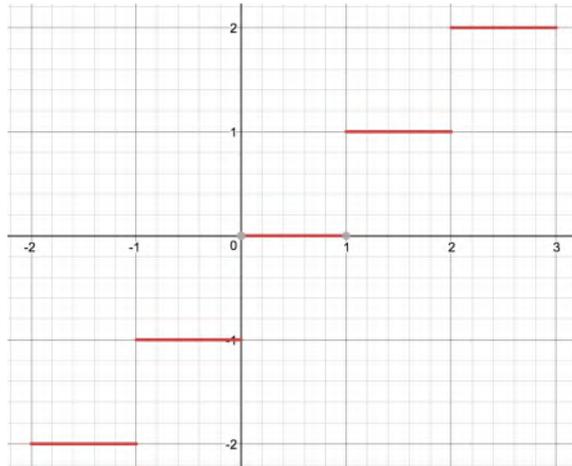
Example 1: Sketch a graph for the following function:

$$f(x) = \lceil x \rceil, \forall x \in [-2, 3)$$

x	$y = \lceil x \rceil$	closed point	open point
$-2 \leq x < -1$	-2	$(-2, -2)$	$(-1, -2)$
$-1 \leq x < 0$	-1	$(-1, -1)$	$(0, -1)$
$0 \leq x < 1$	0	$(0, 0)$	$(1, 0)$
$1 \leq x < 2$	1	$(1, 1)$	$(2, 1)$
$2 \leq x < 3$	2	$(2, 2)$	$(3, 2)$

From the above table, we can see that:

$$D_f = [-2, 3) \text{ and } R_f = \{-2, -1, 0, 1, 2\}$$



Note: In general, $f(x) = [x] = n, \forall n \in \mathbb{I}, \forall x \in [n, n + 1)$ is called “Step Function”.

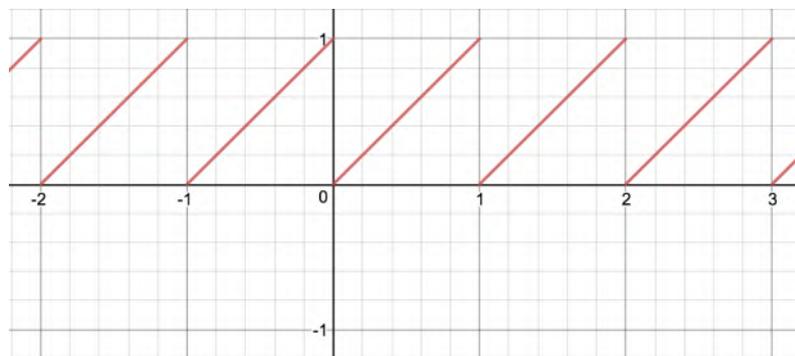
Example 2: Sketch a graph for the following function:

$$f(x) = x - [x], \forall x \in [-3, 3].$$

x	$[x]$	$y = x - [x]$	closed point	open point
$-3 \leq x < -2$	-3	$x + 3$	$(-3, 0)$	$(-2, 1)$
$-2 \leq x < -1$	-2	$x + 2$	$(-2, 0)$	$(-1, 1)$
$-1 \leq x < 0$	-1	$x + 1$	$(-1, 0)$	$(0, 1)$
$0 \leq x < 1$	0	x	$(0, 0)$	$(1, 1)$
$1 \leq x < 2$	1	$x - 1$	$(1, 0)$	$(2, 1)$
$2 \leq x < 3$	2	$x - 2$	$(2, 0)$	$(3, 1)$
$3 = x$	3	$x - 3$	$(3, 0)$	

From the above table, we can see that:

$$D_f = [-3, 3] \text{ and } R_f = \{-3, -2, -1, 0, 1, 2, 3\}$$



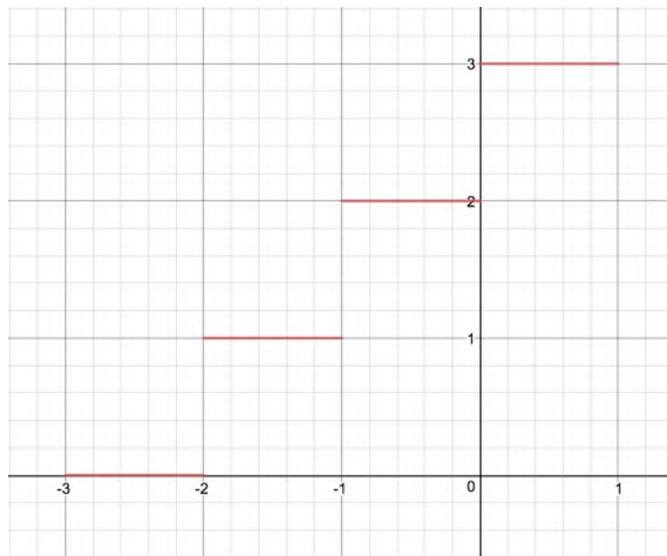
Example 3: Sketch a graph for the following function:

$$f(x) = \lceil 3 + x \rceil, \forall x \in [-3, 1).$$

x	$3 + x$	$y = \lceil 3 + x \rceil$	closed point	open point
$-3 \leq x < -2$	$0 \leq 3 + x < 1$	0	$(-3, 0)$	$(-2, 0)$
$-2 \leq x < -1$	$1 \leq 3 + x < 2$	1	$(-2, 1)$	$(-1, 1)$
$-1 \leq x < 0$	$2 \leq 3 + x < 3$	2	$(-1, 2)$	$(0, 2)$
$0 \leq x < 1$	$3 \leq 3 + x < 4$	3	$(0, 3)$	$(1, 3)$

From the above table, we can see that:

$$D_f = [-3, 1] \text{ and } R_f = \{0, 1, 2, 3\}$$



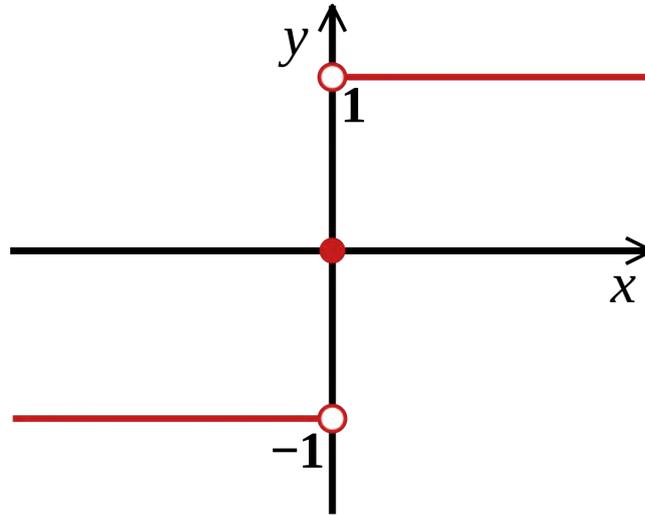
Signum Function:-

We denote the signum function by “ $Sgn(x)$ ”, and it is defined as follows:

$$Sgn(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

$$D_f = \mathbb{R}$$

$$R_f = \{-1, 0, 1\}$$



Example 1: Find the Domain and Range and Sketch a graph for the following function:

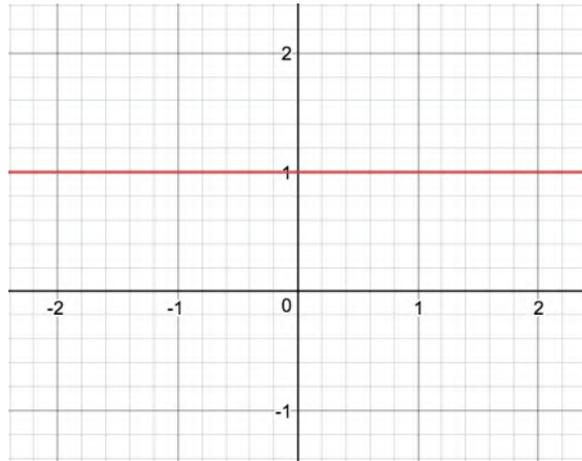
$$f(t) = \text{Sgn}(t^2 + 1)$$

Solution:-

$$f(t) = \text{Sgn}(t^2 + 1) = \begin{cases} 1 & \text{if } t^2 + 1 > 0 \implies t^2 \geq -1 \implies t^2 \geq 0 \implies t \in \mathbb{R} \\ 0 & \text{if } t^2 + 1 = 0 \implies t^2 = -1, \text{ Contradiction} \\ -1 & \text{if } t^2 + 1 < 0 \implies t^2 < -1, \text{ Contradiction} \end{cases}$$

$$\text{Hence, } \text{Sgn}(t^2 + 1) = 1, \forall t \in \mathbb{R}$$

$$D_f = \mathbb{R} \text{ and } R_f = \{1\}$$



Example 2: Find the Domain and Range and Sketch a graph for the following function:

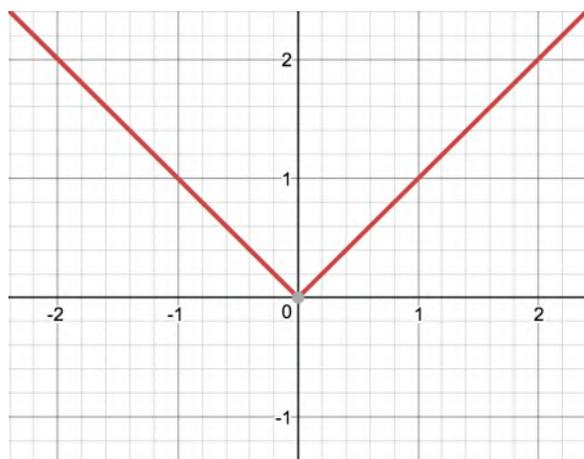
$$g(t) = tSgn(t)$$

Solution:-

$$g(x) = tSgn(t) = t * \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases} = \begin{cases} t & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -t & \text{if } t < 0 \end{cases} = |t|$$

Hence, $g(t) = tSgn(t) = |t|, \forall t \in \mathbb{R}$

$D_f = \mathbb{R}$ and $R_f = \mathbb{R}^+$



Odd Function:-

A function $f(x)$ is called an odd function if $f(-x) = -f(x)$

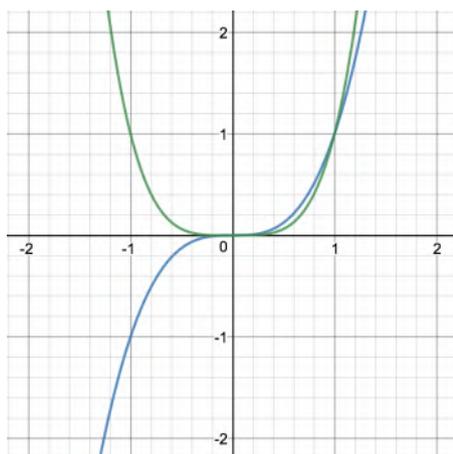
Examples:

- $f(x) = x^3$

$$\because f(-x) = (-x)^3 = -x^3 = -f(x) \implies f(x) \text{ is an odd function.}$$

- $g(x) = x^4$

$$\because g(-x) = (-x)^4 = x^4 \neq -g(x) \implies g(x) \text{ is NOT an odd function.}$$



Note: For odd function, $D_f = R_f = \mathbb{R}$

Even Function:-

A function $f(x)$ is called an Even function if $f(-x) = f(x)$

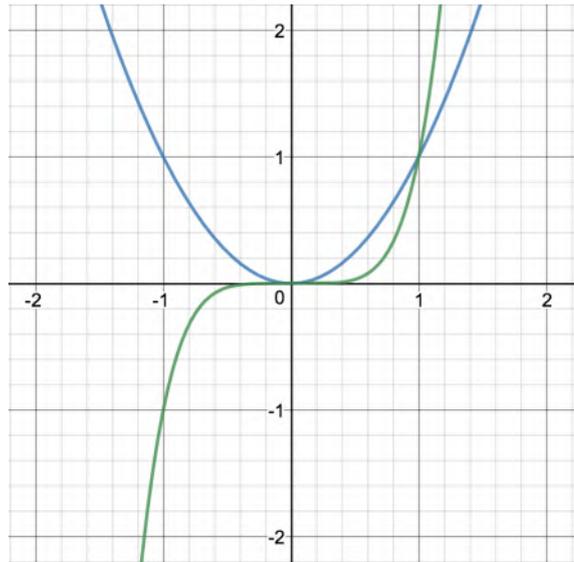
Examples:

- $h(x) = x^2$

$\because h(-x) = (-x)^2 = x^2 = h(x) \implies h(x)$ is an even function.

- $t(x) = x^5$

$\because t(-x) = (-x)^5 = -x^5 \neq t(x) \implies t(x)$ is NOT an even function.



Note: For even function, $D_f = \mathbb{R}$, but $R_f = \mathbb{R}^+$

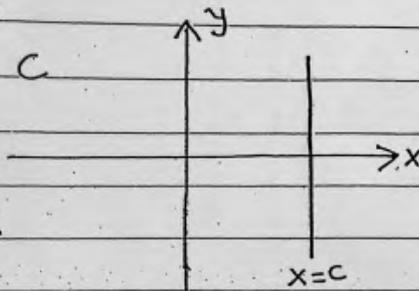
Shifting Function:-

Let $y = f(x)$ s.t. $x \in \mathbb{R}$, and let $c \in \mathbb{R}$, then:

1. $g(x) = f(x) + c$ [Shifting to the **top** c unit]
2. $g(x) = f(x) - c$ [Shifting to the **bottom** c unit]
3. $g(x) = f(x + c)$ [Shifting to the **left** c unit]
4. $g(x) = f(x - c)$ [Shifting to the **right** c unit]
5. $g(x) = -f(x)$ [reflect around x -axes]
6. $g(x) = f(-x)$ [reflect around y -axes]

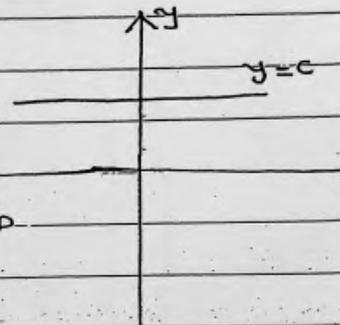
اشكال دوال بصورة كاملة :

{1} $x = c$



خط شاقولي

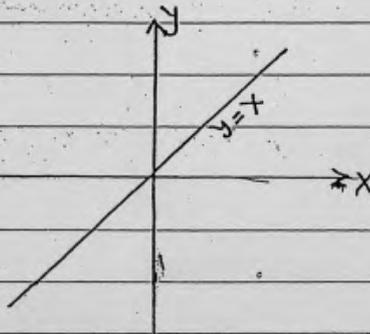
{2} $y = c$



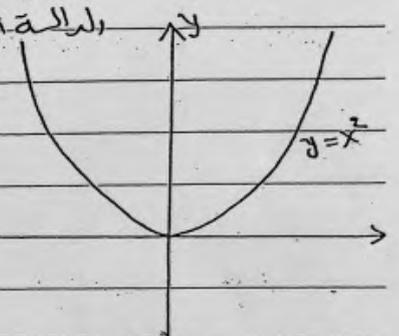
تسمى دالة ثابتة

خط افقي

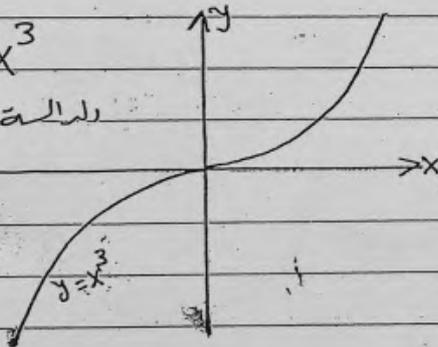
{3} $f(x) = y = x$
دالة خطية



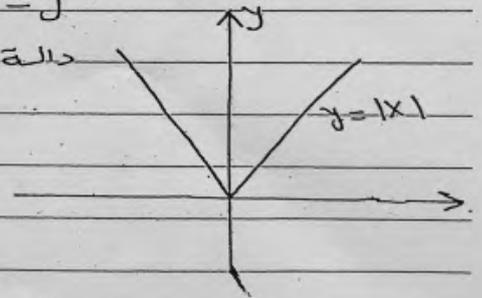
{4} $y = x^2$
الدالة التربيعية



{5} $y = x^3$
الدالة التكعيبية

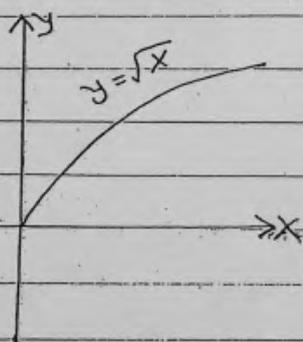


{6} $|x| = y$
دالة المثلث



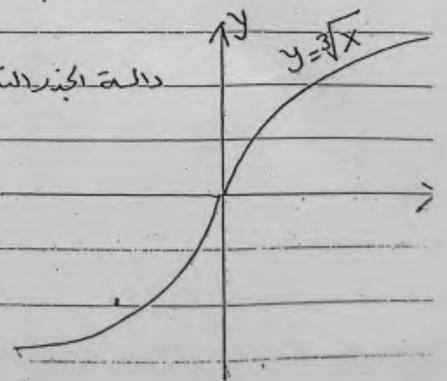
{7} $y = \sqrt{x}$

دالة الجذر التربيعي



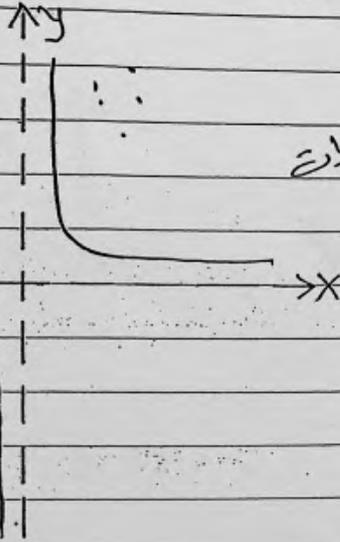
{8} $y = \sqrt[3]{x}$

دالة الجذر التكعيبي



9 $y = \frac{1}{x}$

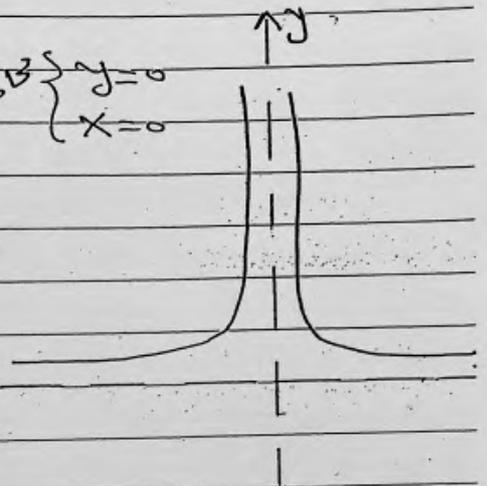
دالة كسرية



10 $y = \frac{1}{x^2}$

مخازنة $\begin{cases} y=0 \\ x=0 \end{cases}$

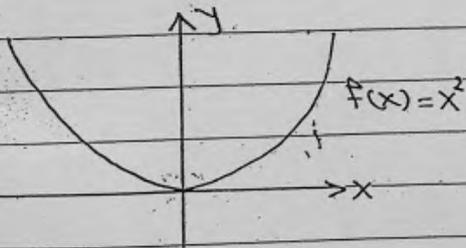
مخازنة $\begin{cases} y=0 \\ x=0 \end{cases}$



مثال (1): افسح فضاء للدالة $f(x) = y = x^2$ ثم جـ:

$g(x) = f(x) + 1$, $k(x) = f(x) - 1$, $h(x) = f(x+1)$, $t(x) = f(x-1)$, $L(x) = f(x)$, $m(x) = f(-x)$

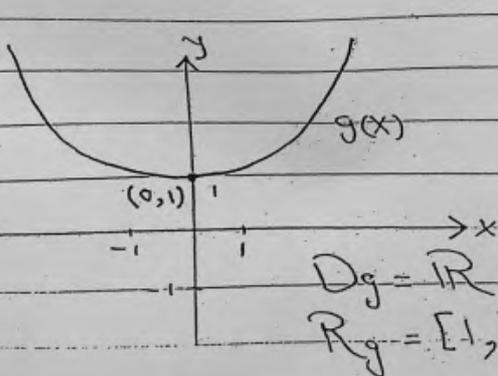
سـ



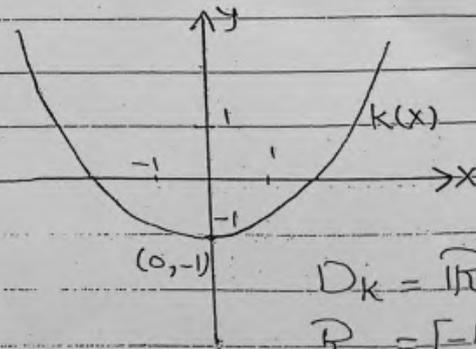
$D_f = \mathbb{R}$
 $R_f = \mathbb{R}^+$
 $= [0, \infty)$

11 $g(x) = f(x) + 1 = x^2 + 1$

12 $k(x) = f(x) - 1 = x^2 - 1$

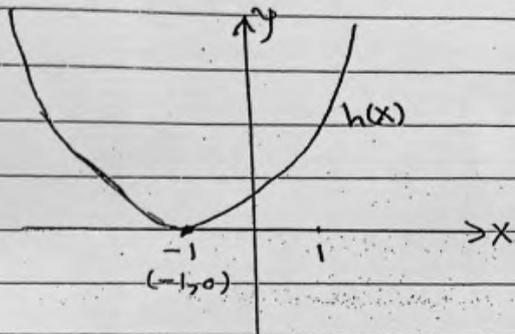


$D_g = \mathbb{R}$
 $R_g = [1, \infty)$



$D_k = \mathbb{R}$
 $R_k = [-1, \infty)$

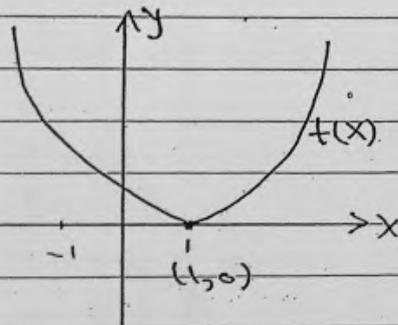
$$\{3\} \quad h(x) = f(x+1) = (x+1)^2$$



$$D_h = \mathbb{R}$$

$$R_h = \mathbb{R}^+ = [0, \infty)$$

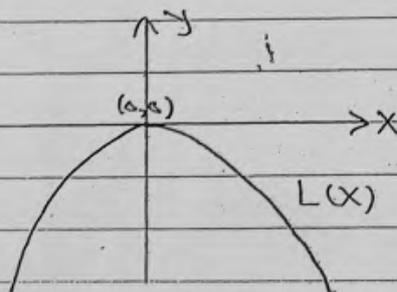
$$\{4\} \quad t(x) = f(x-1) = (x-1)^2$$



$$D_t = \mathbb{R}$$

$$R_t = \mathbb{R}^+ = [0, \infty)$$

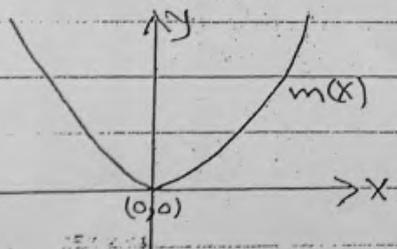
$$\{5\} \quad L(x) = f(x) = -x^2$$



$$D_L = \mathbb{R}$$

$$R_L = \mathbb{R}^- = (-\infty, 0]$$

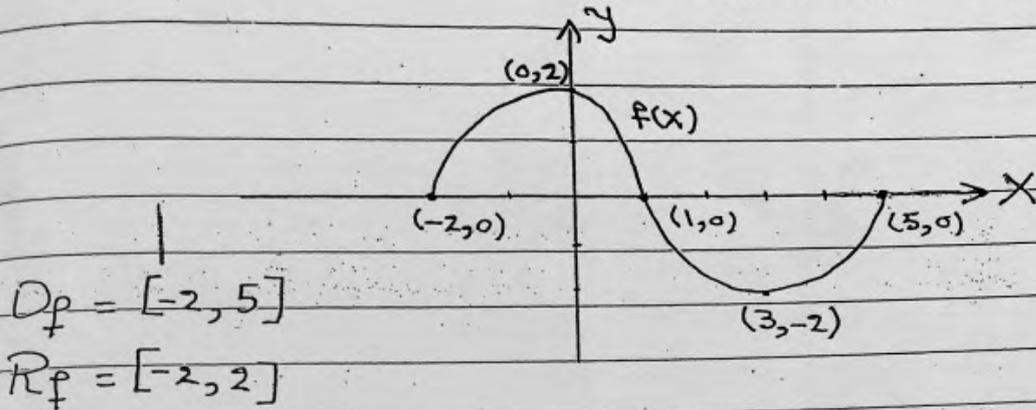
$$\{6\} \quad m(x) = f(-x) = (-x)^2$$



$$D_m = \mathbb{R}$$

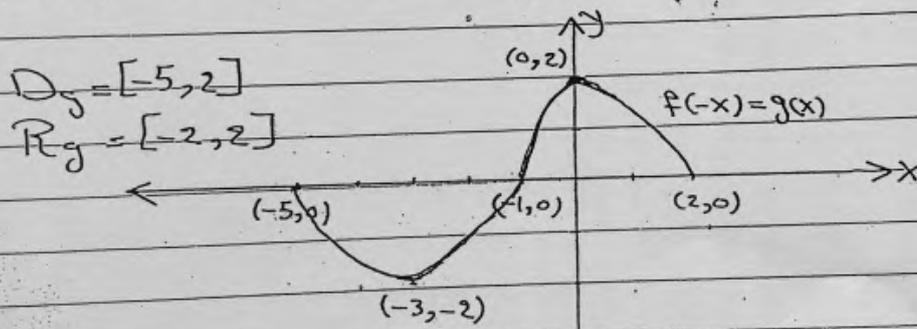
$$R_m = \mathbb{R}^+ = [0, \infty)$$

مثال (5): أرسم منحنى الدالة $f(3-x)$ إذا كانت $f(x)$ معطاه في الشكل المجاور:

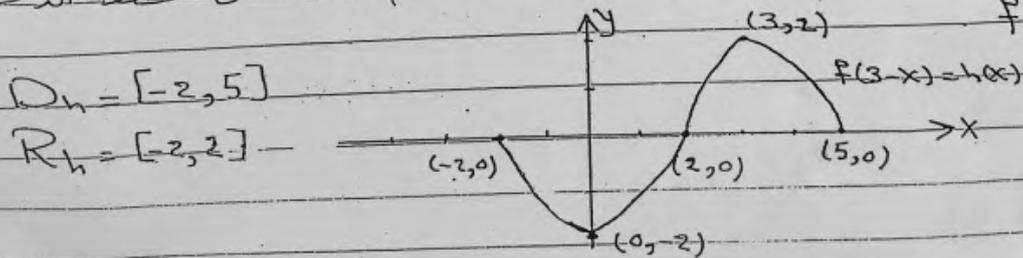


sol

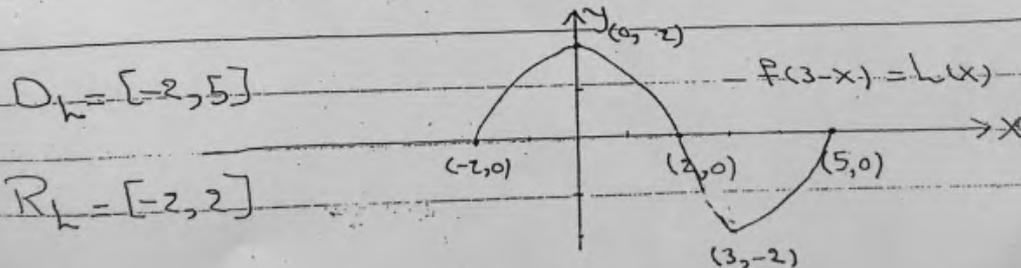
1) نعاكس المنحنى حول محور y للحصول على منحنى $f(-x)$



2) انزاحة المنحنى الى اليمين نأخذ وحداته للحصول على منحنى الدالة $f(3-x)$

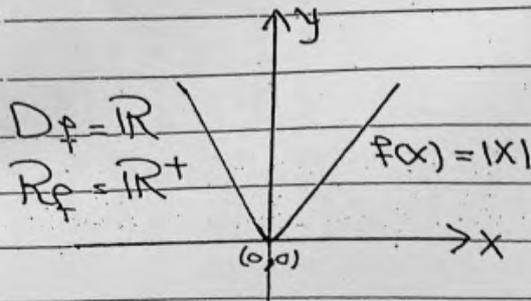


3) نعاكس منحنى الدالة $f(3-x)$ حول محور السينات

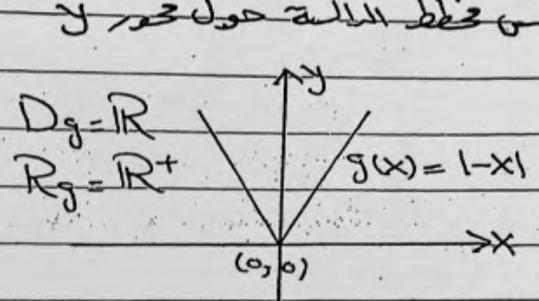


مثال (3): أرسم خط الآلة $y = -|2-x| + 4$

1) $f(x) = |x|$



2) $g(x) = |1-x|$



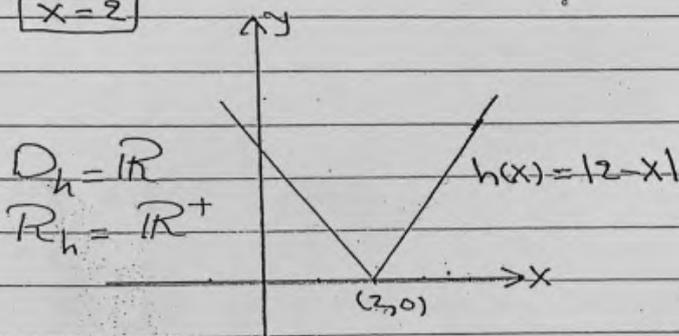
نعكس خط الآلة حول محور y

3) $h(x) = |2-x|$

$2-x=0$

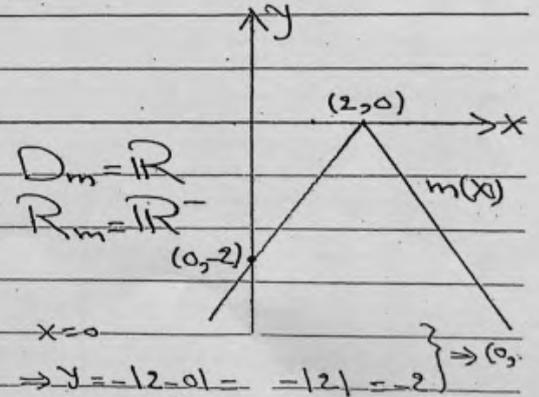
$x=2$

ازاحة نحو اليمين



4) $m(x) = -|2-x|$

نعكس خط الآلة حول محور x



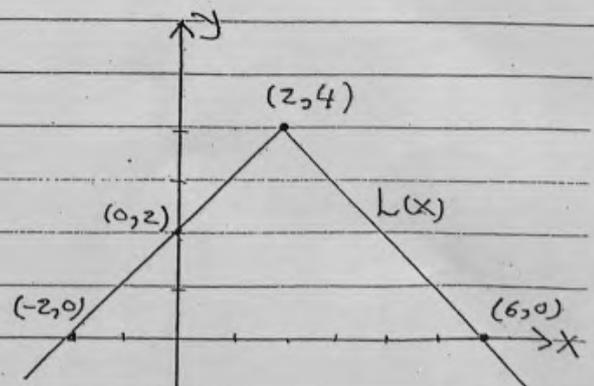
5) $L(x) = -|2-x| + 4$

$D_L = \mathbb{R}$

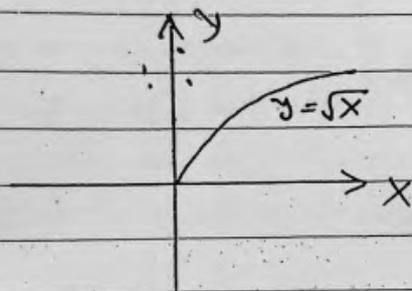
$R_L = (-\infty, 4]$

$x=0 \Rightarrow y = -|2-0| + 4 = 2 \Rightarrow (0,2)$

$y=0 \Rightarrow |2-x|=4 \Rightarrow 2-x=4 \Rightarrow x=-2 \Rightarrow (-2,0)$
 $2-x=-4 \Rightarrow x=6 \Rightarrow (6,0)$



مثال (٤) : الشكل التالي يمثل مخططاً للدالة $y = \sqrt{x}$ ، أكتب
 مخططاً لكل من الدوال الآتية :



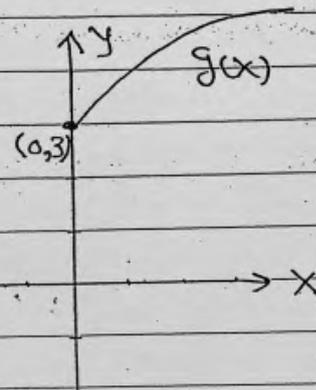
$$D_f = \mathbb{R}^+$$

$$R_f = \mathbb{R}^+$$

1} $g(x) = \sqrt{x} + 3$

$$D_g = \mathbb{R}^+$$

$$R_g = [3, \infty)$$



2} $h(x) = \sqrt{x} - 2$

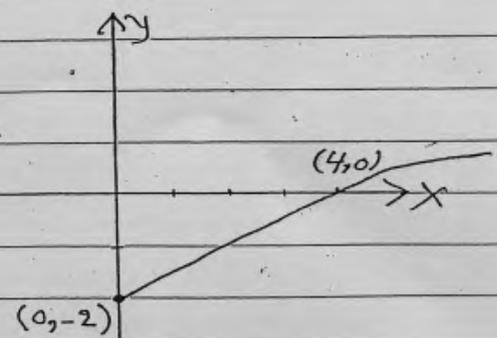
$$y = 0 \Rightarrow 0 = \sqrt{x} - 2$$

$$\Rightarrow \sqrt{x} = 2$$

$$\Rightarrow x = 4 \rightarrow (4, 0)$$

$$D_h = \mathbb{R}^+$$

$$R_h = [-2, \infty)$$

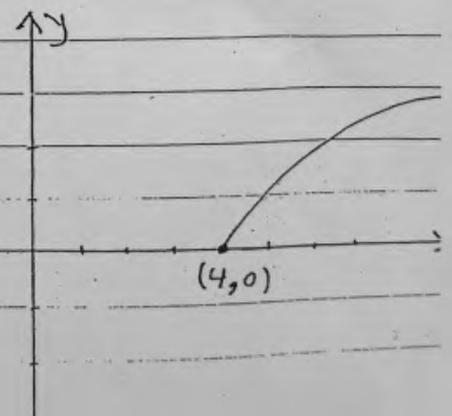


3} $m(x) = \sqrt{x-4}$

$$x-4=0 \Rightarrow x=4$$

$$D_m = [4, \infty)$$

$$R_m = \mathbb{R}^+$$



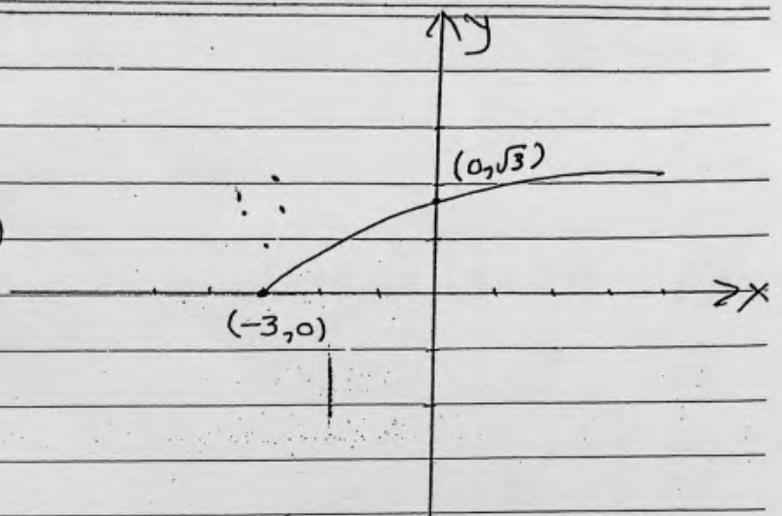
$$4) \sqrt{x+3} = t(x)$$

$$x+3=0 \Rightarrow x=-3$$

$$x=0 \Rightarrow y=\sqrt{3} \Rightarrow (0, \sqrt{3})$$

$$D_f = [-3, \infty)$$

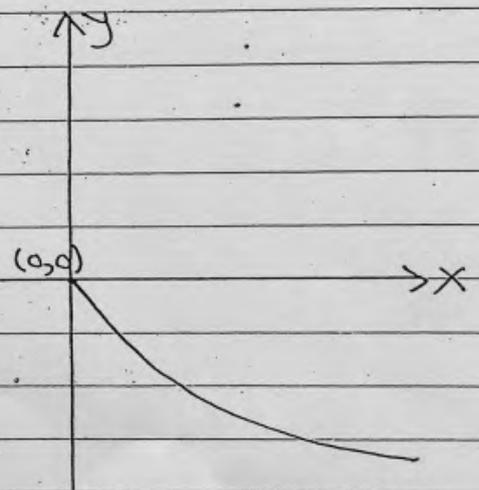
$$R_f = \mathbb{R}^+$$



$$5) h(x) = -\sqrt{x}$$

$$D_h = \mathbb{R}^+$$

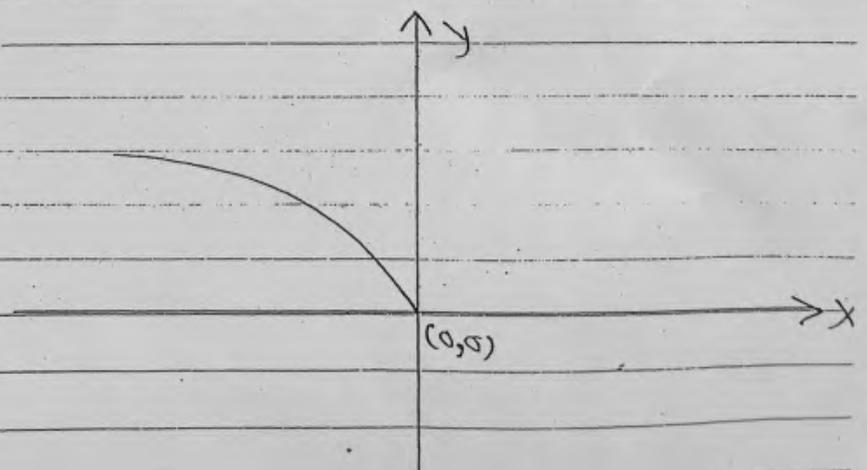
$$R_h = \mathbb{R}^- \\ = (-\infty, 0]$$



$$6) k(x) = \sqrt{-x}$$

$$D_k = \mathbb{R}^- \\ = (-\infty, 0]$$

$$R_k = \mathbb{R}^+$$



CHAPTER THREE: Limits and Continuity

Definition: If the values of $f(x)$ approaches the value L as x approaches c , we say that f has **limit** equal to L as x approaches c , and we write it as:

$$\lim_{x \rightarrow c} f(x) = L$$

Example: Let $f(x) = x^2 + 3$, find the limit of $f(x)$ as x approaches 2.

$x \rightarrow 2^+$ (from the right)	x	3	2.5	2.3	2.1	2.01	2.001	2.0001	
	$f(x)$	12	9.25	8.25	7.44	7.040	7.004	7.0007	$\simeq 7$
$x \rightarrow 2^-$ (from the left)	x	1	1.2	1.4	1.5	1.9	1.99	1.999	
	$f(x)$	4	4.44	4.96	5.95	5.98	6.98	6.999	$\simeq 7$

From the table, we notice that:

- When x approaches 2 from the right, $f(x)$ approaches 7
(i.e., $\lim_{x \rightarrow 2^+} f(x) = 7$).
- When x approaches 2 from the left, $f(x)$ approaches 7
(i.e., $\lim_{x \rightarrow 2^-} f(x) = 7$).

Properties of Limits: Let $\lim_{x \rightarrow c} f_1(x) = L_1$ and $\lim_{x \rightarrow c} f_2(x) = L_2$

where $c, K, L_1, L_2 \in \mathbb{R}$, then:

1. $\lim_{x \rightarrow c} [f_1(x) \mp f_2(x)] = L_1 \mp L_2$

2. $\lim_{x \rightarrow c} [f_1(x) * f_2(x)] = L_1 * L_2$
3. $\lim_{x \rightarrow c} K * f_1(x) = K * L_1$
4. $\lim_{x \rightarrow c} \frac{f_1(x)}{f_2(x)} = \frac{L_1}{L_2}$, where $L_2 \neq 0$

Examples: Evaluate the following limits:

$$1. \lim_{n \rightarrow 5} \frac{\sqrt{4+n}-2}{n} = \frac{\sqrt{4+5}-2}{5} = \boxed{\frac{1}{5}}$$

$$2. \lim_{x \rightarrow 2} \frac{x^2+2x+4}{x+2} = \frac{2^2+2 \cdot 2+4}{2+2} = \frac{12}{4} = \boxed{3}$$

$$3. \lim_{x \rightarrow 5} \frac{x^2-25}{3(x-5)} = \lim_{x \rightarrow 5} \frac{(x+5)(x-5)}{3(x-5)} = \lim_{x \rightarrow 5} \frac{x+5}{3} = \frac{5+5}{3} = \boxed{\frac{10}{3}}$$

$$4. \lim_{h \rightarrow 0} \frac{(2+h)^2-4}{h} = \lim_{h \rightarrow 0} \frac{4+4h+h^2-4}{h} = \lim_{h \rightarrow 0} \frac{h(4+h)}{h}$$

$$= \lim_{h \rightarrow 0} 4 + h = 4 + 0 = \boxed{0}$$

$$5. \lim_{n \rightarrow 0} \frac{\sqrt{4+n}-2}{n} = \lim_{n \rightarrow 0} \frac{\sqrt{4+n}-2}{n} \cdot \frac{\sqrt{4+n}+2}{\sqrt{4+n}+2}$$

$$= \lim_{n \rightarrow 0} \frac{4+n-4}{n(\sqrt{4+n}+2)} = \lim_{n \rightarrow 0} \frac{n}{n(\sqrt{4+n}+2)}$$

$$= \lim_{n \rightarrow 0} \frac{1}{(\sqrt{4+n}+2)} = \frac{1}{\sqrt{4+0}+2} = \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

Right and Left Hand-Side Limits:

Sometimes the value of a function $f(x)$ lead to different limits as x approaches c from different sides.

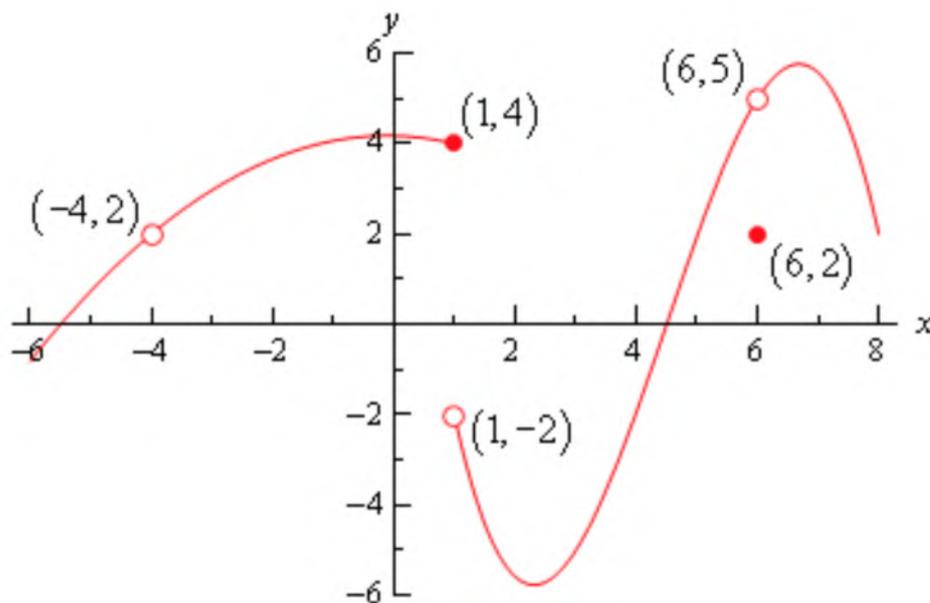
Theorem: Suppose $f(x)$ is defined on an open interval that containing c . Then $\lim_{x \rightarrow c} f(x)$ is defined if and only if $\lim_{x \rightarrow c^+} f(x)$ and $\lim_{x \rightarrow c^-} f(x)$ are both defined and equal.

i.e.,

$$\boxed{\lim_{x \rightarrow c} f(x) = L \iff \lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L}$$

Note If $\lim_{x \rightarrow c^+} f(x) \neq \lim_{x \rightarrow c^-} f(x) \implies \lim_{x \rightarrow c} f(x)$ “**DOES NOT EXIST**”

Example 1: Evaluate the following, where $f(x)$ is defined as shown below.



- $f(6) = 2$
- $\lim_{x \rightarrow 6^-} f(x) = 5$
- $\lim_{x \rightarrow 6^+} f(x) = 5$
- $\implies \lim_{x \rightarrow 6} f(x) = 5$
- $f(1) = 4$
- $\lim_{x \rightarrow 1^-} f(x) = 4$
- $\lim_{x \rightarrow 1^+} f(x) = -2$
- $\implies \lim_{x \rightarrow 1} f(x)$ “Does Not Exist”

Example 2: Let $f(x) = \begin{cases} x^2 - 4 & \text{if } x \leq 3 \\ 5 & \text{if } x > 3 \end{cases}$

Find $\lim_{x \rightarrow 3^+} f(x)$, $\lim_{x \rightarrow 3^-} f(x)$, and $\lim_{x \rightarrow 3} f(x)$

Solution:-

$$\bullet \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^+} \begin{cases} x^2 - 4 & \text{if } x \leq 3 \\ 5 & \text{if } x > 3 \end{cases}$$

$$= \lim_{x \rightarrow 3^+} 5 = 5$$

$$\bullet \lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^-} \begin{cases} x^2 - 4 & \text{if } x \leq 3 \\ 5 & \text{if } x > 3 \end{cases}$$

$$= \lim_{x \rightarrow 3^-} x^2 - 4 = 3^2 - 4 = 9 - 4 = 5$$

$$\bullet \lim_{x \rightarrow 3} f(x) = ?$$

$$\because \lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = 5 \implies \lim_{x \rightarrow 3} f(x) = 5$$

Example 3: Let $g(x) = \begin{cases} \sqrt{x+4} - 1 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \frac{x}{x+3} & \text{if } x > 0 \end{cases}$

Find $\lim_{x \rightarrow 0^+} g(x)$, $\lim_{x \rightarrow 0^-} g(x)$, and $\lim_{x \rightarrow 0} g(x)$

Solution:-

$$\bullet \lim_{x \rightarrow 0^+} g(x) = \lim_{x \rightarrow 0^+} \begin{cases} \sqrt{x+4} - 1 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \frac{x}{x+3} & \text{if } x > 0 \end{cases}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x+3} = \frac{0}{0+3} = 0$$

$$\bullet \lim_{x \rightarrow 0^-} g(x) = \lim_{x \rightarrow 0^-} \begin{cases} \sqrt{x+4} - 1 & \text{if } x < 0 \\ -2 & \text{if } x = 0 \\ \frac{x}{x+3} & \text{if } x > 0 \end{cases}$$

$$= \lim_{x \rightarrow 0^-} \sqrt{x+4} - 1 = \sqrt{0+4} - 1 = 2 - 1 = 1$$

$$\bullet \lim_{x \rightarrow 0} g(x) = ?$$

$\because \lim_{x \rightarrow 0^+} g(x) = 0 \neq 1 = \lim_{x \rightarrow 0^-} g(x) \implies \lim_{x \rightarrow 0} g(x)$ **“DOES NOT EXIST”**

Example 4: Evaluate $\lim_{x \rightarrow 0} |x|$?

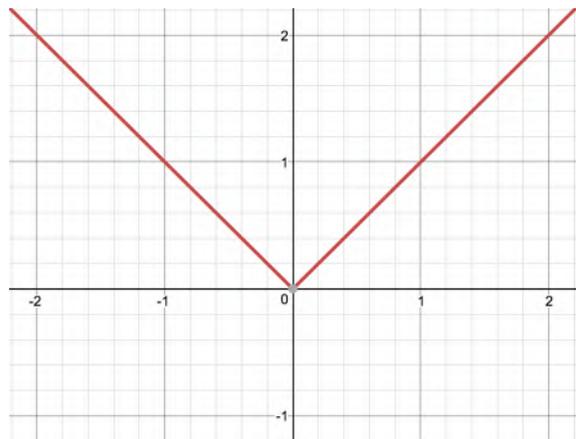
Solution:-

$$\lim_{x \rightarrow 0} |x| = \lim_{x \rightarrow 0} \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

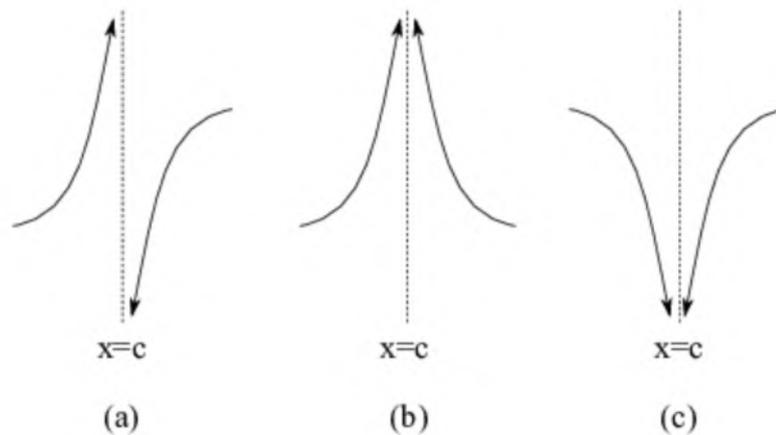
$$\bullet \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^+} (x) = 0$$

$$\bullet \lim_{x \rightarrow 0^-} |x| = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\therefore \lim_{x \rightarrow 0^+} |x| = \lim_{x \rightarrow 0^-} |x| = 0 \implies \lim_{x \rightarrow 0} |x| = 0$$



Infinite ($\mp\infty$) Limits : Let $f(x)$ be defined as follows, then:



In (a): $\lim_{x \rightarrow c^+} f(x) = +\infty$ and $\lim_{x \rightarrow c^-} f(x) = -\infty$

$\implies \lim_{x \rightarrow c} f(x)$ “**DOES NOT EXIST**”

In (b): $\lim_{x \rightarrow c^+} f(x) = +\infty$ and $\lim_{x \rightarrow c^-} f(x) = +\infty$

$\implies \lim_{x \rightarrow c} f(x) = +\infty$

In (c): $\lim_{x \rightarrow c^+} f(x) = -\infty$ and $\lim_{x \rightarrow c^-} f(x) = -\infty$

$\implies \lim_{x \rightarrow c} f(x) = -\infty$

Remark:

$$\boxed{\frac{0}{(\mp)\text{ value}} = 0}$$

$$\boxed{\frac{(+)\text{value}}{0} = +\infty}$$

$$\boxed{\frac{(-)\text{value}}{0} = -\infty}$$

Example 1: Evaluate $\lim_{x \rightarrow 0} \frac{1}{x^2}$?

Solution:-

CHAPTER ONE: The Real Numbers \mathbb{R}

The subsets of \mathbb{R} :

1. **Natural Numbers** (denoted by \mathbb{N}) such that:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

2. **Intager Numbers** (denote by \mathbb{I} or \mathbb{Z}) such that:

$$\mathbb{I} \text{ or } \mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$

3. **Rational Numbers** (denoted by \mathbb{Q}): it is all numbers of the form $\frac{p}{q}$, such that p and q are integers and $q \neq 0$:

$$\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

Example: $\frac{1}{2}, \frac{5}{3}, 0, \frac{50}{10}, \dots$

Note: The rational Numbers can be written as decimal from

$(\frac{1}{3} = 0.333, \frac{1}{4} = 0.25, \dots)$.

4. **Irrational Numbers** (denoted by \mathbb{Q}'): A number which is not rational is said to be irrational.

Example: $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi = 3.14, \dots\}$

Note: $\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

Properties of Real Numbers with Addition: $(\mathbb{R}, +)$

Let $a, b, c \in \mathbb{R}$, then:

1. $a + b \in \mathbb{R}$ (Closure)
 2. $a + b = b + a$ (Commutative)
 3. $a + (b + c) = (a + b) + c$ (Associative)
 4. $a + 0 = 0 + a = a$ (Identity Element)
 5. $\exists(-a) \in \mathbb{R}$ such that $a + (-a) = (-a) + a = 0$ (Additive Inverse)
-

Properties of Real Numbers with Multiplication: (\mathbb{R}, \cdot)

Let $a, b, c \in \mathbb{R}$, then:

1. $a \cdot b \in \mathbb{R}$ (Closure)
 2. $a \cdot b = b \cdot a$ (Commutative)
 3. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (Associative)
 4. $1 \cdot a = a \cdot 1 = a$ (Multiplicative Identity)
 5. $a \cdot (b + c) = a \cdot b + a \cdot c$ (Distributive)
 $(b + c) \cdot a = b \cdot a + c \cdot a$
 6. $\exists a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = a \cdot \frac{1}{a} = 1$ (Multiplication Inverse)
-

Intervals:-

1. **Finite intervals:-** Let $a, b \in \mathbb{R}$ such that $a < b$ then:

(a) **Open Interval** = $\{x \in \mathbb{R} : a < x < b\} = (a, b)$

(Note: $a \notin (a, b)$ and $b \notin (a, b)$)

(b) **Closed Interval** = $\{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$

(Note: $a \in [a, b]$ and $b \in [a, b]$)

(c) **The Half Open Interval** = $\{x \in \mathbb{R} : a < x \leq b\} = (a, b]$

(Note: $b \in (a, b]$ and $a \notin (a, b]$)

OR:

The Half Open Interval = $\{x \in \mathbb{R} : a \leq x < b\} = [a, b)$

(Note: $b \notin [a, b)$ and $a \in [a, b)$)

2. **Infinite intervals:-** Let each of $a, b \in \mathbb{R}$ such that $a < b$ then:

(a) $\{x \in \mathbb{R} \text{ such that } a < x < \infty \text{ (or } x > a) \} = (a, \infty)$

(b) $\{x \in \mathbb{R} \text{ such that } a \leq x < \infty \text{ (or } x \geq a) \} = [a, \infty)$

(c) $\{x \in \mathbb{R} \text{ such that } -\infty < x < a \text{ (or } x < a) \} = (-\infty, a)$

(d) $\{x \in \mathbb{R} \text{ such that } -\infty < x \leq a \text{ (or } x \leq a) \} = (-\infty, a]$

(e) $\{x \in \mathbb{R} \text{ such that } -\infty < x < \infty\} = (-\infty, \infty) = \mathbb{R}$

Inequalities:-

Let $a, b \in \mathbb{R}$, b is **greater** than a and denoted by $b > a$ **if** $b - a > 0$.

Solving Inequalities:-

Solving the inequalities means obtaining all values of x for which the inequality is true.

Properties of Inequalities:-

Let $a, b, c \in \mathbb{R}$, then:

1. if $a < b$, then $a + c < b + c$
2. if $a < b$ and $c > 0$, then $a.c < b.c$
3. if $a < b$ and $c < 0$, then $a.c > b.c$

Note :- In general, we have linear and non-linear inequalities.

Linear Inequalities Examples:-

Example 1: Solve the following inequality: $3(x + 2) < 5$?

solution:-

$$3(x + 2) < 5 \longrightarrow 3(x + 2) < 5 \longrightarrow 3x < 5 - 6 \longrightarrow < \frac{-1}{3}$$

Hence, the solution set $= \{x \in \mathbb{R} : x < \frac{-1}{3}\} = (-\infty, \frac{-1}{3})$.

Example 2: Solve the following inequality: $7 < 2x + 3 < 11$?

solution:-

$$7 < 2x + 3 < 11 \longrightarrow -3 + 7 < 2x < -3 + 11 \longrightarrow 4 < 2x < 8 \longrightarrow 2 < x < 4$$

Hence, the solution set $= \{x \in \mathbb{R} : 2 < x < 4\} = (2, 4)$.

Non-Linear Inequalities Examples:-

Example 1: Solve the following inequality: $x^2 < 25$?

solution:- $x^2 < 25 \rightarrow x^2 - 25 < 0 \rightarrow (x - 5)(x + 5) < 0$

Since the result is negative, then there are two possibilities:

Either:

$$(x + 5) > 0 \text{ and } (x - 5) < 0 \longrightarrow x > -5 \text{ and } x < 5$$

So, the solution set is $(-5, 5)$

Or:

$$(x + 5) < 0 \text{ and } (x - 5) > 0 \longrightarrow x < -5 \text{ and } x > 5$$

So, the solution set is \emptyset

Therefore, the solution set for the inequality is

$$(-5, 5) \cup \emptyset = (-5, 5)$$

Example 2: Solve the following inequality: $x^2 - 5x > 6$?

solution:-

$$x^2 - 5x > 6 \rightarrow x^2 - 5x - 6 > 0 \rightarrow (x - 6)(x + 1) > 0$$

Since the result is Positive, then there are two possibilities:

Either:

$$(x - 6) > 0 \text{ and } (x + 1) > 0 \longrightarrow x > 6 \text{ and } x > -1$$

So, the solution set: $S_1 = \{x \in \mathbb{R} : x > 6\} = (6, \infty)$

Or:

$$(x - 6) < 0 \text{ and } (x + 1), 0 \longrightarrow x < 6 \text{ and } x < -1$$

$$\text{So, the solution set: } S_2 = \{x \in \mathbb{R} : x < -1\} = (-\infty, -1)$$

Therefore, the solution set for the inequality is:

$$S = S_1 \cup S_2 = (6, \infty) \cup (-\infty, -1) = \mathbb{R} \setminus [-1, 6]$$

Absolute Value:-

The absolute value of a real number x is denoted by $|x|$ and defined as follows:

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

Examples: $|-8| = 8$, $|\frac{-2}{3}| = \frac{2}{3}$, $|9| = 9$, $|0| = 0$, etc.

Properties of Absolute Value:-

1. $|-a| = |a|$

proof: $|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$

2. $||a|| = |a|$

proof: $||a|| = \sqrt{|a|^2} = \sqrt{a^2} = |a|$

3. $|a.b| = |a|.|b|$

proof: $|a.b| = \sqrt{(a.b)^2} = \sqrt{a^2.b^2} = \sqrt{a^2}.\sqrt{b^2} = |a|.|b|$

$$4. \left| \frac{a}{b} \right| = \frac{|a|}{|b|}; b \neq 0$$

$$\text{proof: } \left| \frac{a}{b} \right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{|a|}{|b|}$$

$$5. |a + b| \leq |a| + |b|$$

Solving Absolute Value Inequalities:-

The absolute value of x can be written as follows:

$$|x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The above definition means the absolute value of any real number is a real non-negative number.

Geometrically, the absolute value of number x is the distance point between “ x ” and the origin point “0”. In general, $|a - b|$ is the distance between a and b on the real number line “ \mathbb{R} ”.

Remarks:

1. To solve the inequality $|x| < a$ where $a, x \in \mathbb{R}$.

Case (1): If $x \geq 0 \implies |x| = x$,

but $|x| < a \implies x < a. \implies S_1 = (-\infty, a)$

Case (2): If $x < 0 \implies |x| = -x$,

but $|x| < a \implies -x < a \implies x > -a. \implies S_2 = (-a, \infty)$

Since, $S = S_1 \cap S_2$

$$\implies \{x \in \mathbb{R} : |x| < a\} = \{x \in \mathbb{R} : -a < x < a\} = (-a, a)$$

Similarly,

$$\implies \{x \in \mathbb{R} : |x| \leq a\} = \{x \in \mathbb{R} : -a \leq x \leq a\} = [-a, a]$$

2. To solve the inequality $|x| > a$ where $a, x \in \mathbb{R}$.

Case (1): If $x \geq 0 \implies |x| = x$,

but $|x| > a \implies x > a. \implies S_1 = (a, \infty)$

Case (2): If $x < 0 \implies |x| = -x$,

but $|x| > a \implies -x > a \implies x < -a. \implies S_2 = (-\infty, -a)$

Since, $S = S_1 \cup S_2$

$$\implies \{x \in \mathbb{R} : |x| > a\} = (a, \infty) \cup (-\infty, -a) = \mathbb{R} \setminus [-a, a]$$

Similarly,

$$\implies \{x \in \mathbb{R} : |x| \geq a\} = [a, \infty) \cup (-\infty, -a] = \mathbb{R} \setminus (-a, a)$$

Examples:- Find the solution set for the following inequalities?

- $|x| > 3$

solution:-

$$\{x \in \mathbb{R} : |x| > 3\} = \{x \in \mathbb{R} : x > 3 \text{ or } x < -3\} =$$

$$(3, \infty) \cup (-\infty, -3) = \mathbb{R} \setminus [-3, 3]$$

• $|x| \leq 4$

solution:-

$$\{x \in \mathbb{R} : |x| \leq 4\} = \{x \in \mathbb{R} : -4 \leq x \leq 4\} = [-4, 4]$$

• $|x - 4| < 5$

solution:-

$$\{x \in \mathbb{R} : |x - 4| < 5\} = \{x \in \mathbb{R} : -5 < x - 4 < 5\}$$

$$= \{x \in \mathbb{R} : -1 < x < 9\} = (-1, 9)$$

• $|7 - 4x| \geq 1$

solution:-

$$\{x \in \mathbb{R} : |x - 4| \geq 1\} = \{x \in \mathbb{R} : 7 - 4x \geq 1 \text{ or } 7 - 4x \leq -1\}$$

$$= \{x \in \mathbb{R} : -4x \geq -6 \text{ or } -4x \leq -8\}$$

$$= \{x \in \mathbb{R} : x \leq \frac{3}{2} \text{ or } x \geq 2\}$$

$$= (-\infty, \frac{3}{2}] \cup [2, \infty)$$

$$= \mathbb{R} \setminus (\frac{3}{2}, 2)$$

Problems 1.1:

1. Write the following sets equivalent interval, and test of these intervals whether they are Open, Close or Half Open Intervals:

(a) $\{x : -20 \leq x \leq -12\}$

(c) $\{x : -1 < x < 10\}$

(b) $\{x : -3 \leq x < 4\}$

(d) $\{x : -2 < x \leq 0\}$

2. Give a description of the following intervals as sets:

- (a) $(3, 5)$ (c) $[2, 7]$ (e) $(-4, 4)$
(b) $(-3, 0)$ (d) $[-5, -1)$ (f) $(-0, 7]$

3. Find the solution set of the following inequalities:

- (a) $x(x - 3) > 4$ (h) $6x - 4 > 7x + 2$
(b) $2 < \frac{1}{x}; x \neq 0$ (i) $x^2 \leq 16$
(c) $x^2 \geq 25$ (j) $3x^2 > 2x + 5$
(d) $x^2 - 2x - 24 < 0$ (k) $x^2 > 5x + 6$
(e) $-7 \leq -3x + 5 \leq 14$ (l) $\frac{x-3}{x+2} < 5$
(f) $\frac{x}{x-3} < 4$ (m) $\frac{1}{x-2} > \frac{2}{x+3}$
(g) $\frac{x^2+2x-35}{x+2} > 0$ (n) $\frac{x-2}{x+3} < \frac{1}{2}$

4. Find the solution set of the following inequalities:

- (a) $|x| \geq 5$ (g) $\frac{|2-x|}{3x} \leq 1$
(b) $|x| < 2$ (h) $|\frac{3+2x}{3x}| \leq 1$
(c) $|3x + 3| \geq 2$ (i) $|x - 1| \geq 6$
(d) $1 \leq |\frac{x-3}{1-2x}| \leq 2$ (j) $|2 - 2x| \leq 7$
(e) $|\frac{2-x}{x-3}| \geq 4$ (k) $|\frac{4}{2x+1}| \leq 3$
(f) $|x + 1| < |3x + 4|$



CHAPTER TWO: The Functions

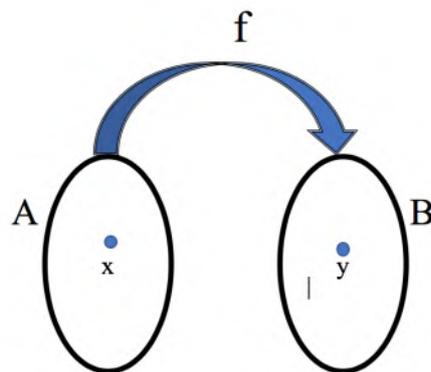
Definition: Let A and B be two non-empty sets, the relation that assigns to every element $x \in A$, with a unique value $y \in B$ is called a **function**. i.e.,

$$f : A \longrightarrow B; \forall x \in A \exists! y \in B \text{ such that } f(x) = y$$

Notes:

1. $A = \text{Domain} = D_f$

$$B = \text{Co-domain} = \text{Co-} D_f$$

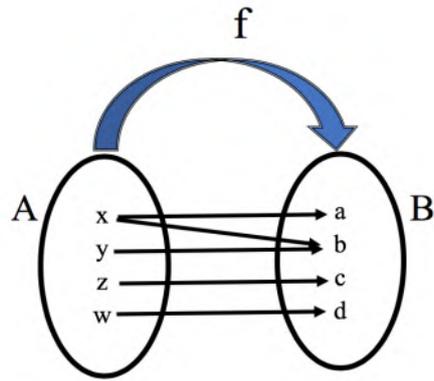


2. The set of all images $f(x) = y, \forall x \in D_f$ is called the Range of f .

i.e., $R_f = \{f(x) = y; x \in D_f\}$

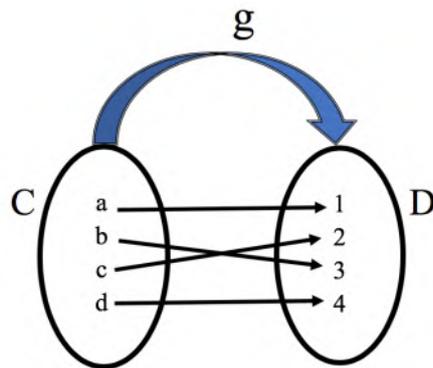
Functions and Non-functions Examples:-

- **Example(1):**



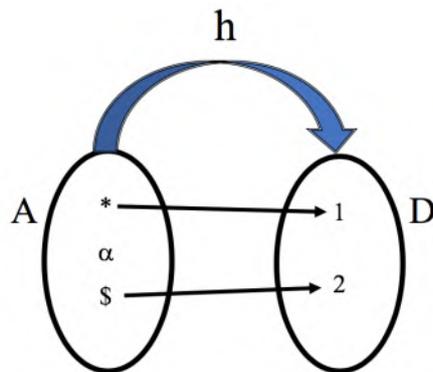
f is not a function because $f(x) = a$ and $f(x) = b$
 (i.e., x has two images).

• **Example(2):**



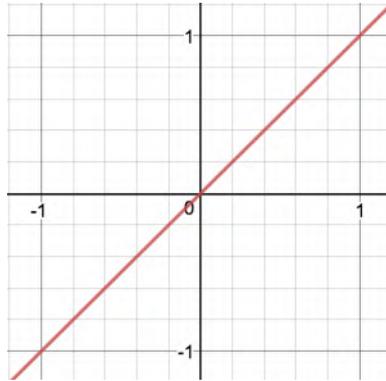
g is a function and $R_g = \{1, 3, 4\}$.

• **Example(3):**



h is not a function because $\alpha \in A$ and α has not image.

- **Example(4):** $y = x$ is a Linear function.

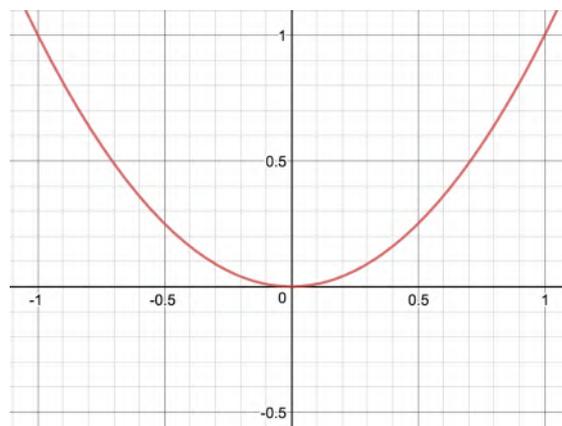


$$y = f(x) = x, f : \mathbb{R} \longrightarrow \mathbb{R}$$

$$D_f = \mathbb{R} = \{x \in \mathbb{R} : -\infty < x < \infty\}$$

$$R_f = \mathbb{R} = \{y \in \mathbb{R} : -\infty < y < \infty\}$$

- **Example(5):** $y = x^2$ is a Non-linear function.

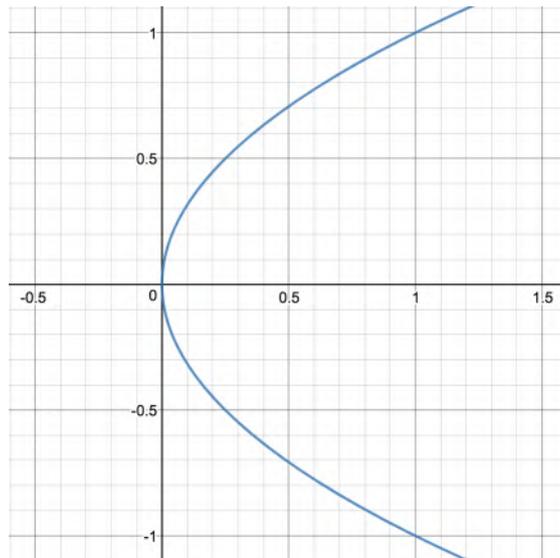


$$y = f(x) = x^2, f : \mathbb{R} \longrightarrow [0, \infty]$$

$$D_f = \mathbb{R}$$

$$R_f = \mathbb{R}^+ = \{y \in \mathbb{R} : y \geq 0\} = [0, \infty)$$

- **Example(6):** Is $y^2 = x$ a function?

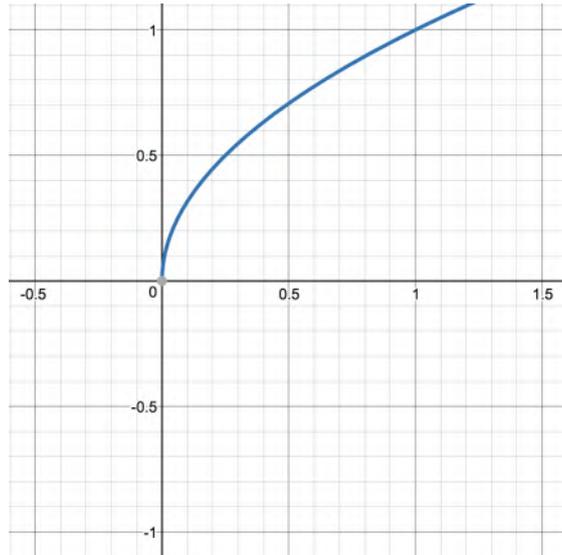


$$y^2 = x \longrightarrow \sqrt{y^2} = \sqrt{x} \longrightarrow |y| = \mp \sqrt{x}$$

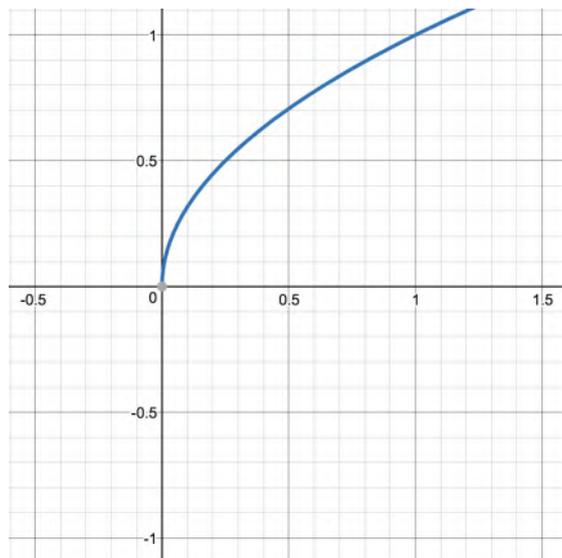
$\forall x \in D_f, \exists \mp \sqrt{x}$ (i.e., there are two images for each x).

Hence, “ $y^2 = x$ ” is not a function.

However, $y_1 = +\sqrt{x}$ is a function.



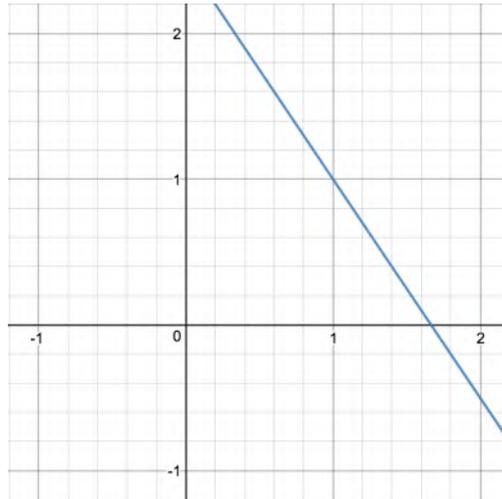
Also, $y_2 = -\sqrt{x}$ is a function.



-
- **Example(7):** Is $2y + 3x = 5$ a function?

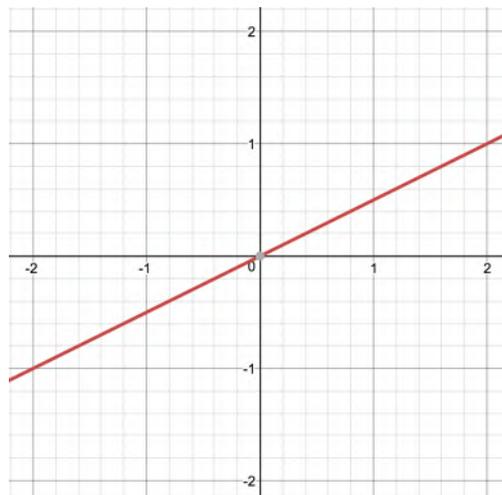
$$2y + 3x = 5 \longrightarrow 2y = 5 - 3x \longrightarrow y = \frac{5-3x}{2}$$

Since for each value of x there exists only one value of y , it is a function.



-
- **Example(8):** Is $\frac{x}{y} = 2$ a function?

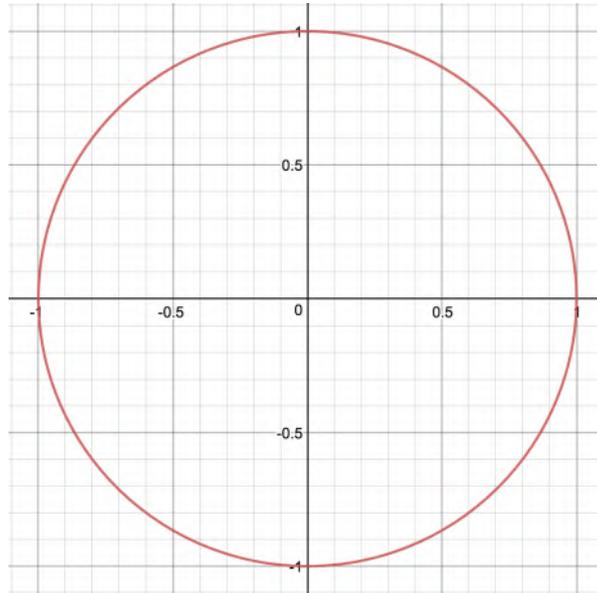
$$\frac{x}{y} = 2 \implies y = \frac{1}{2}x$$



Since for each value of x there exists only one value of y , it is a function.

- **Example(9):** Is $x^2 + y^2 = 1$ a function?

$$x^2 + y^2 = 1 \longrightarrow y = \mp\sqrt{1 - x^2}$$

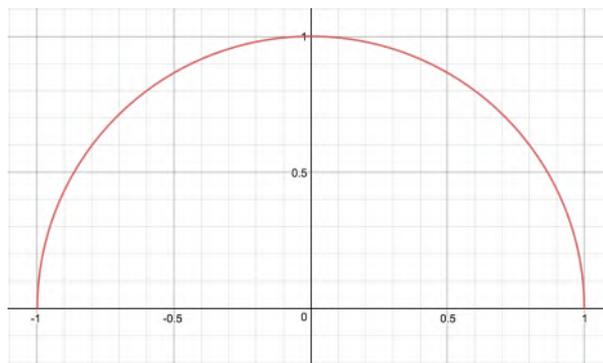


Since $\forall x \in D_f, \exists \mp\sqrt{1 - x^2}$ (i.e., there are two images for each value of x), “ $x^2 + y^2 = 1$ ” is not a function.

However, $y_1 = f_1(x) = +\sqrt{1 - x^2}$ is a function.

$$1 - x^2 \geq 0 \longrightarrow x^2 \leq 1 \longrightarrow -1 \leq x \leq 1$$

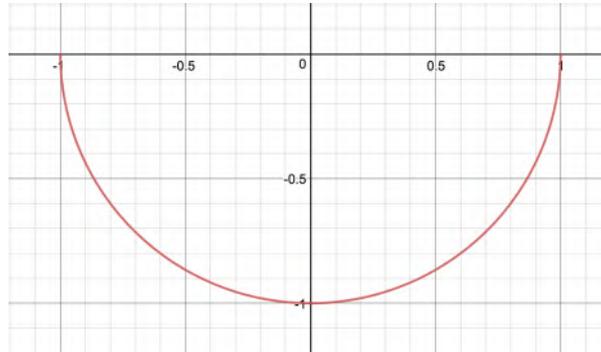
$$D_{f_1} = [-1, 1] \text{ and } R_{f_1} = [0, 1]$$



Also, $y_2 = f_2(x) = -\sqrt{1 - x^2}$ is a function.

$$1 - x^2 \geq 0 \longrightarrow x^2 \leq 1 \longrightarrow -1 \leq x \leq 1 \text{ (same the above)}$$

$$D_{f_1} = [-1, 1] \text{ and } R_{f_1} = [-1, 0]$$



How to Find the Domain and the Rang of a Function?

Remark (1): The domain of all polynomials or odd roots is all real numbers.

Example: Find the domain and the rang of the following functions?

1. $f(x) = x^3 + 2x^2 + 3x - 5$

$$D_f = \mathbb{R}; R_f = \mathbb{R}$$

2. $g(x) = \sqrt[3]{x^7 - 1}$

$$D_g = \mathbb{R}; R_g = \mathbb{R}$$

Remark (2): The domain of even root is all the real numbers such that the expression under the radical is greater than or equal to zero.

Example 1 : Let $f(x) = \sqrt{x^2 - 4}$, find D_f and R_f ?

To find D_f :

$$x^2 - 4 \geq 0 \implies (x - 2)(x + 2) \geq 0$$

$$\text{either: } x - 2 \geq 0 \wedge x + 2 \geq 0 \implies x \geq 2 \wedge x \geq -2 \implies [2, \infty)$$