Foundation Of Mathematics

أسس الرياضيات

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قسم الرياضيات

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Foundation of Mathematics

أسس الرياضيات

Foundations of mathematics is the study of the basic mathematical concepts (logic statements العبارات المنطقية, numbers, relations, sets, functions...).

أسس الرياضيات هو علم دراسة المفاهيم الرياضية الأساسية كالعبارات المنطقية ، أنظمة الأعداد ، العلاقات ، المجاميع والدوال.

<u>Set of Numbers (Subsets of the set of real numbers $\mathbb{R}$)</u>

1. Set of Natural numbers $N = \{1,2,3, \dots\}$

2. Set of Prime numbers $P = \{2,3,5,7,11, \dots\}$

3. Set of Integer numbers $Z = I = \{\dots, -2, -1,0,1,2, \dots\}$

4. Set of Even numbers $E = \{\dots, -4, -2,0,2,4, \dots\}$

5. Set of Odd numbers $O = \{\dots, -3, -1,1,3, \dots\}$

6. Set of Rational numbers $Q = \{\frac{a}{b} : a, b \in Z, b \neq 0\}$

<u>Example:</u> $\frac{2}{3}, -\frac{1}{5}, 3, 0.5, 0.3333$ are examples of rational numbers

7. Set of Irrational numbers $H = \{x: x \notin \mathrm{Q}\}$

<u>Example:</u> $\pi = 3.1415$ .... is irrational number

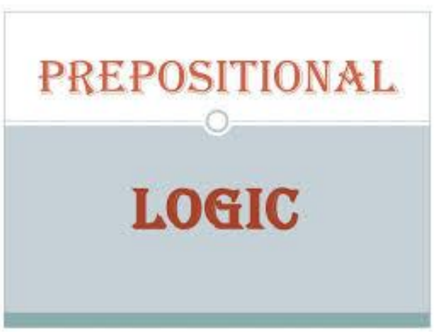
$e = 2.71828 \dots$ is irrational number

$\sqrt{2}, \sqrt{5}$ are irrational numbers

CHAPTER ONE

Mathematical Logic and Proof Using Propositional Calculus

المنطق والبرهان الرياضي باستخدام العبارات الخبرية



Chapter One Contents

1. **Propositions (Statements) العبارات**
2. **Compound Propositions العبارات المركبة**
3. **Mathematical proof البرهان الرياضي**
4. **Quantifiers المسورات**

<u>Definition1.1</u>: Mathematical Logic المنطق الرياضي

Mathematical logic is a subfield of mathematics exploring the applications of formal logic to mathematics. Mathematical logic is widely used in theoretical computer science and other sciences.

المنطق الرياضي هو احد الحقول الرياضية التي تدرس تطبيقات المنطق في الرياضيات. المنطق الرياضي يستخدم بشكل واسع في علوم الحاسبات وعلوم أخرى.

<u>Propositions or Statements</u> العبارات

<u>Definition</u>1.2: A **proposition** is a declarative sentence which is either '**true: T**' or '**false: F**', but not both. We use the letters $p, q, r, s, ..etc$ to denote a proposition.

العبارة هي **<u>جملة خبرية</u>** والتي قد تكون صادقة أو كاذبة ومن غير الممكن أن تكون العبارة صادقة وكاذبة بنفس الوقت.

<u>Example1.3:</u> Which of the following sentences are called propositions (statements), and which ones are not propositions.

1) p: $\sqrt{4} = 2$ is a **true proposition**

2) q: $\sum_{x=1}^{3} (x+2) = 13$ is a **false proposition**

Because $\sum_{x=1}^{3} (x+2)$ = (1+2) + (2+2) + (3+2) =3+4+5= 12 ≠ 13.

3) r: Baghdad isn't in Iraq is a **false statement**

4) s: What time is it? is **not a proposition**

لأنها جملة استفهامية وليست جملة خبرية

5) w: Study hard is **not a proposition**

Because it is not a declaration sentence ليست جملة خبرية

6) v: $x + y = 0$ is **not a proposition**

لان الجملة ليست صادقة ولا كاذبة

<u>Example1.4:</u> (**H. W**) Which of the following sentences is called a proposition (statement), and which one is not a proposition.

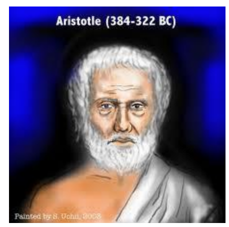
أي من الجمل أدناه تمثل عبارة وأيها لاتمثل عبارة؟

i) $p\colon\ x+1=3$

ii) $q\colon\ x+y\ =\ z$

iii) $r\colon\ \frac{3}{4}$ is an even number (عدد زوجي)

The area of logic that deals with propositions is called **propositional logic** or **propositional calculus.** It was first developed by the Greek philosopher **Aristotle** أرسطو more than 2300 years ago.



<u>Definition1.5</u>: Negation of a proposition نفي العبارة

Let p be a proposition. The ***negation*** of p is called ‘‘not p’’ and is denoted by ($\sim p$).

لتكن ‘p’ عبارة. يقال للعبارة ‘not p’ أو ‘ليس p’ أنها نفي العبارة p ويرمز لها بالرمز $\sim p$

<u>Example1.6:</u> Re-write the following expressions without using the negation

$\sim(3<5)$

$\sim(x>y)$

$\sim(x\geq 5)$

$\sim(2=10)$

Example1.7: Find the truth value of each of the following statements. Find $(\sim p)$ negations for the statements q and r.

أوجد قيم صدق ونفي كل من العبارات التالية:

1. p : Today is Saturday (F) , $\sim p$: Today is **not** Saturday
2. q : 2+2=4 **(T)**

   $\sim q: 2+2 \neq 4$

3. r : The square has four sides **(H. W)**

Remark1.8: If a proposition p is true, then $\sim p$ is false; and if p is false, then $\sim p$ is true.

أدناه جدول صدق العبارة p ونفيها

The truth table of the negation of a proposition *p*	
p	$\sim p$
T	F
F	T

Double Negation Law: If p is a proposition, then $\sim\sim p = p$.

p	$\sim p$	$\sim\sim p$
T	F	**T**
F	T	**F**

Compound propositions

Propositions are divided into two types:

1. **Primitive proposition** **عبارة بدائية أو بسيطة :** A proposition is said to be **primitive,** if it cannot be divided into simpler propositions.

   العبارة تسمى بدائية إذا لم يمكن تحليلها إلى عبارات أبسط .

2. **Composite proposition** **عبارة مركبة:** A proposition is called **composite**, if it is compound of more than one primitive proposition using logical connective operators.

العبارة تسمى مركبة إذا كانت تتكون من عبارتين بسيطتين أو أكثر تربطها أداة ربط واحدة او اكثر.

ادوات الربط التي تكون العبارة المركبة هي: and $\wedge$

or $\vee$

if then $\Rightarrow$

if and only if $\Leftrightarrow$

Example1.9: Propositions **(1)-(3)** in Example (1.3) are primitive.

Example1.10: The following propositions are composite:

1. ‘‘ 2+3 = 5 **and** 6-4 = 1’’

   عبارة مركبة مكونة من عبارتين بسيطة مربوطة بأداة الربط **and**

2. ‘‘Ali is clever **or** he studies every day’’

   عبارة مركبة مكونة من عبارتين بسيطة مربوطة بأداة الربط **or**

ملاحظة: قيمة صدق العبارة المركبة تعتمد على:

١. قيم صدق العبارات البسيطة المكونة لها

٢. أدوات الربط المستخدمة لربط العبارات البسيطة

<u>Basic Logical connective Operators</u> أدوات الربط المنطقية الأساسية

There are some basic logical operators that connect simple propositions to produce composite proposition. These operators are:

1. <u>Conjunction</u> operator (و) أداة الوصل--English word (**and**), symbol (**∧**).

Let p and q are two primitive propositions. The conjunction of p and q is denoted by ‘‘$p \wedge q$’’ and read as ‘‘p and q’’.

If both p and q are true, then $p \wedge q$ is true, otherwise $p \wedge q$ is false.

أي عبارتين بسيطتين p و q يمكن ربطهما بأداة الربط (و) لتكوين العبارة المركبة ‘‘p ∧q’’. إذا كانت كل من p و q صادقة فان $p \wedge q$ تكون صادقة. إذا كانت إحدى العبارتين على الأقل كاذبة فان $p \wedge q$ تكون كاذبة.

Below is the truth table for the conjunction of two propositions:

Conjunction		
p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

<u>Example1.11</u>: Find the truth value of the following statements:

أوجد قيم صدق العبارات التالية

1. $2 + 2 = 4$ and $2 + 3 = 5$

   T ∧ T = T

2. $\frac{x}{x} = 1$ such that $x \neq 0 \wedge$ Baghdad is not in Iraq

   T ∧ F= F

3. -5 is a prime number $\wedge\ \pi$ is a rational number

F ∧ F= F

Example1.12: Let p: $\mathrm{x}+\mathrm{y}=\mathrm{y}+\mathrm{x}$ such that $x, y \in N$

q: $2 > 10$

r: There are three seasons in Iraq

Find the truth value of:

i) $(q \wedge \sim r) \wedge r$,

ii) $(q \wedge \sim\sim q) \wedge (\sim \mathrm{p} \wedge \mathrm{r})$

Solution of (i): $\sim r$: The seasons in Iraq are not three.

$$(q \wedge \sim \mathrm{r}) \wedge r = (F \wedge T) \wedge F = F \wedge F = F$$

Example1.13: Let p and q are two propositions such that

p: Fouad is poor (T)

q: Fouad is clever (T)

Find the conjunction of p and q. Discuss the truth values of ''p **and** q''.

اوجد عبارة الوصل بين p و q. ناقش قيم صدق العبارة 'p and q' .

Solution: The conjunction ''p **and** q'' is

''Fouad is poor **and** Fouad is clever''

The compound proposition ''p **and** q'' is **true** if

''Fouad is poor and Fouad is clever''

The compound proposition $(p \wedge q)$ is **false** if:

''Fouad is rich ∧ Fouad is clever''

''Fouad is poor ∧ Fouad is not clever''

''Fouad is not rich ∧ Fouad is not clever''

<u>Properties of the conjunction operators:</u> **خواص أداة الوصل ($\wedge$)**

Let p, q and r are three propositions. Using the truth table, show that:

1. $p \wedge q = q \wedge p$ (commutative خاصية الإبدال)
2. $(p \wedge q) \wedge r = p \wedge (q \wedge r)$ (associative خاصية التجميع) **(H.W.)**
3. $p \wedge p = p$ (Idempotent lawقانون تساوي القوى)
4. $p \wedge T = p$ (Identity law)
5. $p \wedge F = F$ (Domination Law)
6. $p \wedge \sim p = F$

<u>Solution</u>:

1. $p \wedge q = q \wedge p$

p	q	$p \wedge q$	$q \wedge p$
T	F	**F**	**F**
T	T	**T**	**T**
F	T	**F**	**F**
F	F	**F**	**F**

3. $p \wedge p = p$

p	p	$p \wedge p$
T	T	**T**
F	F	**F**

4. $p \wedge F = F$

p	F	$p \wedge F$
T	F	**F**
F	F	**F**

<u>2. Disjunction operator</u> (أو) أداة الفصل--English word (**or**), symbol ($\vee$).

Let p and q be two propositions. The disjunction of p and q is denoted by ‘‘$p \vee q$’’ and read ‘‘p or q’’.

We say that ‘‘$p \vee q$’’ is true when p is true **or** q is true **or both** are true. If both p and q are false, then $p \vee q$ is false.

إذا كانت كل من p و q عبارتين بسيطة. تكون العبارة المركبة ‘‘$p \vee q$’’ المرتبطة بأداة الفصل ($\vee$) كاذبة إذا كانت كل من العبارة p و q عبارة كاذبة. وتكون العبارة

‘‘$p \vee q$’’ صادقة فيما عدا ذلك (أي إذا كانت إحدى العبارتين البسيطتين على الأقل صادقة).

Disjunction		
p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

The truth table for the disjunction of two propositions

<u>Example1.14</u>: (H. W) Let p, q and r are three propositions such that

p: dogs can fly

$q: x - x = 0,\ x \in \mathbb{R}$

$r: -3 \in N$

Find the truth value of the following statements:

a) $(p \vee q) \vee r$

b) $\sim q \vee r$

c) $\sim(\sim p \vee q)$

d) $(p \wedge q) \vee (q \vee r)$

<u>Solution of (c)</u>:

$\sim(\sim p \vee q) = \sim(T \vee T) = \sim T = F$

<u>Example1.15</u>: Let p and q are two primitive propositions such that

p: Today is Friday (T)

q: It is raining today (T)

What is the disjunction of the propositions p and q? Discuss the truth value of ‘‘$p \lor q$’’.

<u>Solution</u>: The disjunction ‘‘$p \lor q$’’ is

‘‘Today is Friday or it is raining today’’

‘‘$p \lor q$’’ means that today is **either** Friday **or** raining **or** both.

The compound proposition ($p \lor q$) is **false** if:

‘‘Today is not Friday or it is not raining today’’

The compound proposition ($p \lor q$) is **true** if:

‘‘Today is Friday or it is raining today’’

‘‘Today is not Friday or it is raining today’’

‘‘Today is Friday or it is not raining today’’

<u>Properties of the disjunction operator:</u> **($\lor$) خواص أداة الفصل**

Let p, q and r are three propositions. Using the **truth table**, show that:

1. $p \lor q = q \lor p$ (خاصية الإبدال) **(H. W)**
2. $(p \lor q) \lor r = p \lor (q \lor r)$ (خاصية التجميع)
3. $p \lor p = p$ (قانون تساوي القوى) **(H. W)**
4. $p \lor T = T$ (Domination Law) **(H. W)**
5. $p \lor F = p$ (Identity Law) **(H. W)**
6. $p \lor \sim p = T$ **(H. W)**

<u>Solution2</u>: $(p \lor q) \lor r = p \lor (q \lor \mathrm{r})$

p	q	r	$p \lor q$	$q \lor \mathrm{r}$	$(p \lor q) \lor r$	$p \lor (q \lor \mathrm{r})$
T	T	T	T	T	**T**	**T**
F	F	F	F	F	**F**	**F**
F	T	T	T	T	**T**	**T**
F	T	F	T	T	**T**	**T**
F	F	T	F	T	**T**	**T**
T	F	F	T	F	**T**	**T**
T	T	F	T	T	**T**	**T**
T	F	T	T	T	**T**	**T**

3. <u>Conditional operator</u> أداة الشرط: English word (**if…then**), Arabic word (**إذا كان فان**), symbol ($\rightarrow$).

Let p and q be two propositions. The conditional statement ‘‘$p \rightarrow q$’’ is the proposition ‘‘if p then q’’. The conditional statement ‘‘$p \rightarrow q$’’ is **false** when p is true and q is false, otherwise ‘‘$p \rightarrow q$’’ is **true**.

إذا كانت كل من p و q عبارة بسيطة فان العبارة المركبة (if…then)

يرمز لها بالرمز ($\rightarrow$).

تكون العبارة $(\text{if } p \text{ then } q)$ **كاذبة** في حالة واحدة فقط عندما تكون p عبارة صادقة و q عبارة كاذبة. تكون العبارة $(\text{if } p \text{ then } q)$ **صادقة** فيما عدا ذلك.

ملاحظة: إذا كانت p عبارة خاطئة فان العبارة '$\text{if } p \text{ then } q$' تكون غير قابلة للاختبار وبالتالي فأنها لايمكن أن تكون خاطئة.

The following is the truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

<u>Remark</u>1.16: In the conditional statement ''$p \to q$'', p is called the **hypothesis** فرضية and q is called the **conclusion** **نتيجة** .

<u>Remark</u>1.17: The conditional statement can be expressed in the following equivalent ways:

a) ''p implies q''

b) ''q if p'' ,

c) ''q only if p'',

d) ''p is sufficient condition for q'',

e) ''q is a necessary condition for p''.

<u>Example1.18</u>: Find the truth value of the following statements:

1. If fish fly, then $3 + 2 = 5$

   $F \to T = T$

2. If fish walk, then $3 + 2 = 6$

   $F \to F = T$

<u>Example1.19</u>: Let p, q, and r are three propositions such that

p: 3 is an odd number

$q: x + y = y + x,\ x, y\ \in \mathbb{R}$

r: Winter is hot

Find the truth value of the following statements:

1) $(p\ \rightarrow\ q) \lor (r\ \rightarrow q)$ **(H. W)**
2) if $(p\ \land\ q)$ then $(q\ \lor\ \sim r)$
3) $(p\ \land\ r) \lor (q \rightarrow p)$ **(H. W)**

<u>Solution2</u>:

if $(p\ \land\ q)$ then $(q\ \lor\ \sim r)$

if $(T\ \land\ T)$ then $(T\ \lor\ T)$

if T then $T =\ T$

<u>Example1.20</u>: Find the truth value of the following statements:

1. The statement: ''**If** *x* is negative **then** − *5x* is positive''

$$\mathrm{T} \rightarrow T = T$$

2. The statement: "**If** 9 > 5 **then** dogs don't fly"

$$\mathrm{T} \rightarrow T = T$$

3. The statement: ''**If** $(x\ >\ 0$ and $x^2 < 0)$ **then** $x \geq 10$''

If (T and F**) then** $F(or\ T)$

If F **then** $F(or\ T) = T$

4. The statement: ''**If** $x\ >\ 0$ **then** $(x^2\ <\ 0$ or $2x\ <\ 0)$''

$$\mathrm{T} \rightarrow (F\ or\ F) =\ T\ \rightarrow F = F$$

<u>Definition 1.21</u>: Let p and q are two propositions, then

1. The proposition ‘‘$q \ \rightarrow p$’’ is called the **converse** of ‘‘$p \ \rightarrow \ q$’’.

The proposition ‘‘$\sim$p $\rightarrow \sim q$’’ is called the **inverse** of ‘‘$p \ \rightarrow \ q$’’.

<u>Example1.22</u>: What is the converse and the inverse of the conditional statement:

‘‘**if** $x \ > \ 5, x \ \in N$ **then** $x \ > \ 3$’’?

What is the truth value of the statement and its inverse and converse?

Solution: The statement ‘‘**if** $x \ > \ 5$ **then** $x \ > \ 3$’’

The truth value: T $\rightarrow$ T= T

The **converse** is ‘‘**if** $x \ > \ 3$ **then** $x \ > \ 5$’’

The truth value: for $x = \{4,5\}$; T $\rightarrow$ F= F

For $x = \{6,7, \ldots \ldots\}$, }; T $\rightarrow$ T= T

The **inverse** is **if** $x \ \leq \ 5$ **then** $x \ \leq \ 3$’’

The truth value: for $x = \{1,2,3\}$; T $\rightarrow$ T= T

For $x = \{4,5\}$, }; T $\rightarrow$ F= F

<u>Properties of the conditional operator:</u> **($\rightarrow$) خواص أداة الشرط**

Let p, q and r are three propositions. Using the **truth table** show that (**H. W**.)

1. $p \ \rightarrow \ q \ \neq$ q $\rightarrow \ p$
2. $(p \ \rightarrow \ \ q) \rightarrow r \ \neq p \ \rightarrow \ (q \ \rightarrow$ r)
3. **Find** the truth value of: $p \ \rightarrow \ T, \ p \ \rightarrow \ F, \ p \rightarrow \sim p, \ p \ \rightarrow p$

4. Bi-conditional operator أداة الشرط المزدوج: English word (if and only if)**,** Arabic word (إذا وفقط إذا), **symbol** ($\leftrightarrow$)

Let p and q be propositions. The *bi-conditional* statement ‘‘$p \leftrightarrow q$’’ is the proposition ‘‘p if and only if q’’. The bi-conditional statement is **true** when p and q have the same true value and is **false** otherwise.

لتكن كل من **p** و **q** عبارة بسيطة. تكون العبارة المركبة ’’**p** إذا وفقط إذا **q**‘‘ والتي يرمز لها بالرمز ($\leftrightarrow$) صادقة في حالة تشابه قيم صدق العبارتين وكاذبة فيما عدا ذلك.

T	T	T
T	F	F
F	T	F
F	F	T

The truth table for the bi-conditional of two propositions

<u>**Remark1.23**</u>**:** There are some other ways to express ‘‘$p \leftrightarrow q$’’:

‘‘p iff q’’

‘‘if p then q , and if q then p’’

‘‘p is necessary and sufficient for q’’.

<u>**Example1.24**</u>**:** Find the truth value of ‘‘$x > 0 \leftrightarrow 2x > 0$’’

<u>**Solution**</u>: The statement is true because

If $x > 0$ then $2x > 0$ and if $2x > 0$ then $x > 0$

<u>**Example1.25**</u>**:** Find the truth value of ‘‘$x > 0 \leftrightarrow x^2 > 0$’’ (**H. W.**)

Example1.26: Let p: you can take the flight (True)

q: you can buy a ticket (True)

Then the bi-conditional statement $p \leftrightarrow q$ is

‘‘you can take the flight if and only if you can buy a ticket’’

Discuss the truth values of the bi-conditional statement.

Solution: The statement $p \leftrightarrow q$ is **true**

‘‘ you buy a ticket ↔ can take the flight’’

or

‘‘you do not buy a ticket ↔ cannot take the flight’’.

The statement $p \leftrightarrow q$ is **false** when p and q have opposite truth values.

‘‘you do not buy a ticket ↔ you can take the flight’’

or

‘‘you buy a ticket ↔ cannot take the flight’’.

Properties of the biconditional operator خواص أداة الشرط المزدوج

Let p, q and r are three propositions. Using the **truth table** show that: (**H. W**)

1. $p \leftrightarrow q = q \leftrightarrow p$
2. $(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$
3. **Find** the truth value of: $p \leftrightarrow T, p \leftrightarrow F, p \leftrightarrow \sim p, p \leftrightarrow p$.

Exercise1.27:

1. Find the truth value of the following statements:

[(if $2 + 3 = 4$ then $x + 4 = 4 + x$) and 8 is an even number] iff ($2 \leq -10$ or $2 \geq -10$).

Solution: $[(F \rightarrow T) \wedge T] \leftrightarrow (F \vee T) = [T \wedge T] \leftrightarrow T = T \leftrightarrow T = T$

2. Let p: horse can swim

q: Conjunction operator is useful

r: $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ for $x, y \in N$

Find the truth value of the following statements:

1. $p \leftrightarrow r$
2. $(p \rightarrow r) \wedge q$
3. $[(p \rightarrow r) \vee (q \rightarrow \sim p)]$

3. Write the truth table of the following statements:

i) $\sim p \wedge q$

ii) $(p \wedge q) \rightarrow (p \vee q)$

iii) $(p \rightarrow q) \vee \sim (q \leftrightarrow p)$

4. Write the following statements using the connections operators $\rightarrow, \leftrightarrow, \wedge, \vee$

i) If p and q integer numbers and $q \neq 0$ then $\frac{p}{q}$ is a rational number

ii) If x^2 is integer number then x is even or odd number

iii) $xy > 0$ if and only if ($x > 0$ and $y > 0$) or ($x < 0$ and $y < 0$)

Definition1.28: A compound proposition that is always true is called a ***tautology*** or ***lemma*** or ***theorem*.**

يقال للعبارة المركبة التي تكون صادقة دائما بأنها **نظرية** أو **تحصيل حاصل**

A compound proposition that is always false is called a ***contradiction***.

يقال للعبارة المركبة والتي تكون خاطئة دائما بأنها **تناقض**.

<u>Example1.29</u>: Show that $(p \vee \sim p)$ is tautology and $(p \wedge \sim p)$ is contradiction.

<u>Solution</u>:

p	$\sim p$	$p \vee \sim p$ tautology	$p \wedge \sim p$ contradiction
T	F	T	F
F	T	T	F

<u>Example1.30:</u> (H. W) which of the following compound statements is theorem (tautology) and which one is contradiction

$$p \wedge F, \quad p \vee T, \quad p \leftrightarrow \sim p, \quad [(p \to q) \wedge p] \wedge \sim q$$

<u>Definition1.31</u>: Logical Equivalence التكافؤ المنطقي

Two statements (propositions) that have **same** truth values are called **logically equivalent**. The notation $p \equiv q$ or $p = q$ denotes that p and q are logically equivalent.

تكون عبارتين متكافئة منطقيا إذا كان لهما نفس قيمة الصدق ويرمز للتكافؤ المنطقي بالرمز ≡ أو =

<u>Example1.32</u>: show that $\sim (p \vee q) = \sim p \wedge \sim q$ (logically equivalent).

Hint: make truth table

<u>Solution</u>: The truth table for $\sim (p \vee q)$ and $\sim p \wedge \sim q$ is

p	q	$p \vee q$	$\sim (p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	**F**	F	F	**F**

T	F	T	**F**	F	T	**F**
F	T	T	**F**	T	F	**F**
F	F	F	**T**	T	T	**T**

<u>Definition</u>1.33: Let p, q and r three propositions, then define the following logical equivalence:

1. $p \wedge q = \sim(\sim p \vee \sim q)$
2. $p \rightarrow q = \sim p \vee q$
3. $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

ملاحظة: التعريف أعلاه مهم جدا ويجب أن يحفظ

<u>De Morgan's Theorem:</u> Let p and q are two propositions. Then

1. $\sim(p \wedge q) = \sim p \vee \sim q$ **(H. W)**

2. $\sim(p \vee q) = \sim p \wedge \sim$q

<u>Proof (2)</u>: Take the **right-hand side** (**R. H. S**)

$\sim p \wedge \sim q = \sim(\sim\sim p \vee \sim\sim q)$ [definition of $\wedge$]

$= \sim (p \vee q)$ [double negation law: $\sim\sim p = p$]

= **Left hand side (L. H. S)**.

<u>Exercise1.34:</u> Simplify the following statements:

1. $\sim(p \vee \sim q)$

2. $\sim(\sim p \rightarrow q)$

3. $\sim(\sim p \leftrightarrow q)$

<u>Solution</u> (1): $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$ [De Morgan's law]

$= \sim p \wedge q$ $[\sim\sim q = q]$

<u>Solution</u> (2): $\sim(\sim p \rightarrow q) = \sim(\sim\sim p \vee q)$

$= \sim(p \vee q)$ $[\sim\sim p = p]$

$= \sim p \wedge \sim q$ [De Morgan's law]

<u>Laws of Logical Equivalence</u> قوانين التطابق المنطقي

Let p, q and r are propositions. The following are some of the common logical equivalence rules:

1. Commutative Law قانون الإبدال: $p \wedge q = q \wedge p$

$$p \vee q = q \vee p$$

$$p \leftrightarrow q = q \leftrightarrow p$$

2. Associative Law قانون التجميع: $(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$$(p \vee q) \vee r = p \vee (q \vee r)$$

$$(p \leftrightarrow q) \leftrightarrow r = p \leftrightarrow (q \leftrightarrow r)$$

3. Distributive Law (from left) قانون التوزيع من اليسار:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \wedge r) = (p \wedge q) \wedge (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

$$p \vee (q \vee r) = (p \vee q) \vee (p \vee r)$$

$$p \vee (q \to r) = (p \vee q) \to (p \vee r)$$

$$p \vee (q \leftrightarrow r) = (p \vee q) \leftrightarrow (p \vee r)$$

4. Distributive Law (from right) قانون التوزيع من اليمين:

$$(q \vee r) \wedge p = (q \wedge p) \vee (r \wedge p)$$

$$(q \wedge r) \wedge p = (q \wedge p) \wedge (r \wedge p)$$

$$(q \wedge r) \vee p = (q \vee p) \wedge (r \vee p)$$

$$(q \vee r) \vee p = (q \vee p) \vee (r \vee p)$$

$$(q \to r) \vee p = (q \vee p) \to (r \vee p)$$

$$(q \leftrightarrow r) \vee p = (q \vee p) \leftrightarrow (r \vee p)$$

5. Idempotent Law قانون تساوي القوى : $p \wedge p = p$; $p \vee p = p$

6. Identity Law: $p \wedge T = p$; $p \vee F = p$

7. Domination Law: $p \wedge F = F$; $p \vee T = T$

<u>Exercise1.35:</u> Simplify the following statements using laws of logical equivalence:

بسط العبارات التالية باستخام قوانين التطابق المنطقي

1. $(p \vee q) \wedge \sim p$

2. $(p \vee q) \vee (\sim p \wedge q)$

<u>Exercise1.36:</u> Prove (**without using the truth table**) that

برهن بدون استخدام جداول الصدق

$$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim q$$

<u>Solution</u>: Take the **L. H. S**

$\sim (p \vee (\sim p \wedge q)) = \sim p \wedge \sim(\sim p \wedge q)$ [De Morgan's law]

$= \sim p \wedge (\sim\sim p \vee \sim q)$ [De Morgan's law]

$= \sim p \wedge (p \vee \sim q)$ [by double negation law]

$= (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ [by distributive law]

$= F \vee (\sim p \wedge \sim q)$ $[\sim p \wedge p = F]$

$= (\sim p \wedge \sim q) \vee F$ [by commutative law]

$= \sim p \wedge \sim q$ **R. H. S**

<u>Theorem1.37</u>: (Properties of →)

Let p, q and r are three propositions. Prove the following properties **without using truth tables**:

1. $p \to p = T$
2. $\sim p \to p = p$
3. $p \to T = T$
4. $T \to p = p$
5. $p \to F = \sim p$
6. $F \to p = T$
7. $p \to q = \sim q \to \sim p$
8. $p \to q = (p \wedge \sim q) \to \sim p$

9. $p \to q = (p \wedge \sim q) \to (r \wedge \sim r)$

10. $\sim(p \to q) = p \wedge \sim q$

<u>Proof 1</u>: To prove $p \to p = T$

$$p \to p = \sim p \vee p \quad \text{[def. of } \to]$$

$$= T$$

<u>Proof 4</u>: To prove $T \to p = p$

$$T \to p = \sim T \vee p \quad \text{[def. of } \to]$$

$$= F \vee p \quad [\sim T = F]$$

$$= p$$

<u>Proof 7</u>: To prove $p \to q = \sim q \to \sim p$

$$p \to q = \sim p \vee q \quad \text{[def. of } \to]$$

$$= q \vee \sim p \quad [\text{v is commutative}]$$

$$= \sim q \to \sim p$$

<u>Proof 8</u>: To prove $p \to q = (p \wedge \sim q) \to \sim p$

Take the R. H. S: $(p \wedge \sim q) \to \sim p$

$$= \sim(p \wedge \sim q) \vee \sim p \quad \text{[def. of } \to]$$

$$= (\sim p \vee \sim\sim q) \vee \sim p \quad [\text{ De Morgan}]$$

$$= (\sim p \vee q) \vee \sim p \quad [\sim\sim q = q]$$

$$= \sim p \vee (q \vee \sim p) \quad [\vee \text{ is associative}]$$

$$= \sim p \vee (\sim p \vee q) \quad [\vee \text{ is comm.}]$$

$$= (\sim p \vee \sim p) \vee q \quad [\vee \text{ is asso.}]$$

$$= \sim p \vee q \quad [\, p \vee p = p]$$

$$= p \to q \quad [\text{def. of} \to] = \text{L.H.S}$$

<u>Theorem1.38</u>: (Properties of ↔)

Let p and q are two propositions. Prove the following properties without using truth tables:

1. $p \leftrightarrow p = T,\ p \leftrightarrow T = p,\ p \leftrightarrow F = \sim p$

2. $p \leftrightarrow \sim p = F$

3. $p \leftrightarrow q = q \leftrightarrow p$

4. $p \leftrightarrow q = \sim p \leftrightarrow \sim q$

5. $\sim p \leftrightarrow q = p \leftrightarrow \sim q$

6. $\sim(p \leftrightarrow q) = \sim p \leftrightarrow q$

7. $\sim(p \leftrightarrow q) = p \leftrightarrow \sim q$

<u>Proof 1</u>: To prove $p \leftrightarrow T = p$

$$p \leftrightarrow T = (p \to T) \wedge (T \to p) \quad [\text{ def. of} \leftrightarrow]$$

$$= (\sim p \vee T) \wedge (\sim T \vee p) \quad [\text{ def. of} \to]$$

$$= (\sim p \vee T) \wedge (F \vee p) \quad [\sim T = F]$$

$$= T \wedge p \quad [\sim p \vee T = T]$$

$$= p$$

<u>Proof 6</u>: $\sim(p \leftrightarrow q) = \sim p \leftrightarrow q$

Take **L. H. S** : $\sim(p \leftrightarrow q) = \sim[(p \to q) \wedge (q \to p)]$ [def. of ↔]

$$= \sim(p \to q) \vee \sim(q \to p) \quad [\text{ De Morgan}]$$

$$= \sim(\sim p \vee q) \vee \sim(\sim q \vee p) \quad [\text{ def. of} \to]$$

$= (p \wedge \sim q) \vee (q \wedge \sim p)$ [De Morgan]

$= [(p \wedge \sim q) \vee q] \wedge [(p \wedge \sim q) \vee \sim p)]$ [distributive ($\vee$ on $\wedge$)]

$= [(p \vee q) \wedge (\sim q \vee q)] \wedge [(p \vee \sim p) \wedge (\sim q \vee \sim p)]$ [dist. ($\vee$ on $\wedge$)]

$= [(p \vee q) \wedge T] \wedge [T \wedge (\sim q \vee \sim p)]$

$= (p \vee q) \wedge (\sim q \vee \sim p)$

$= (\sim p \to q) \wedge (q \to \sim p)$ [def. of $\to$] $= \sim p \leftrightarrow q$ [def. of $\leftrightarrow$]

<u>Mathematical Proof</u> البرهان الرياضي

A mathematical proof is a valid argument that establishes the truth of a mathematical statement.

البرهان الرياضي هو إثبات صحة عبارة رياضية من خلال حجة أو تعليل منطقي.

<u>Methods of Proving Mathematical Statements (or Theorems)</u>

1. <u>Direct Proof of a conditional statement $p \to q$</u>

Direct proofs lead from the hypothesis of a theorem to the conclusion.

<u>Definition1.39</u>: The integer number x is called **even** if there exist $k \in Z$ such that $x = 2k$.

<u>Definition1.40</u>: The integer number x is called **odd** if there exist $k \in Z$ such that $x = 2k + 1$.

<u>Theorem1.41</u>: If x is an odd natural number ($x \in$ O) then x^2 is odd

<u>Proof</u>: Assume that x is an odd natural number. We must prove x^2 is odd

Since x is odd, then $x = 2k + 1$ for some $k \in N$.

$x^2 = x.x = (2k + 1)(2k + 1) = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

Let $s = 2k^2 + 2k \in N$, then $x^2 = 2s + 1$

Hence, x^2 is an odd number.

Theorem1.42: (**H. W.**) If x is an even natural number ($x \in$ E) then x^2 is even.

Theorem1.43: The sum of two even natural numbers is even

The theorem can be written as follows: If $x, y \in E^+$ then $x + y \in E^+$ where $E^+ =$ set of positive even numbers.

Proof: Let p: x *and* y are even positive numbers,

q: $x + y$ is an even positive number

Let $x = 2r$ and $y = 2s$ ($r, s \in N$). Then $x + y = 2r + 2s = 2(r + s)$ such that $r + s \in N$

$x + y = 2k$ where $k = r + s$. Therefore $x + y$ is a positive even number.

Theorem1.44: (H. W.)

i) If $x \in E$ and $y \in O$ then $x + y \in O$

ii) If $x \in E$ and $y \in O$ then $xy \in E$

iii) If $x, y \in E$ then $x + y \in E$

2. Direct Proof of a biconditional statement $p \leftrightarrow q$

To prove a proposition in the form $p \leftrightarrow q$, we prove its equivalence. i.e.,

$$\boldsymbol{p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)}$$

Theorem1.45: x is odd number $\leftrightarrow x + 1$ is an even number

Proof: Let p: x is odd number

q: $x + 1$ is an even number

من التعريف أعلاه يجب أن نبرهن بان **p → q** و **q → p**

1. **Prove $p \to q$:** Let $x \in O$, $x = 2k + 1\ ; k \in Z$

$x + 1 = 2k + 2 = 2(k + 1); \quad (k + 1) \in Z$

$x + 1 = 2r \ ; r = k + 1 \in Z$

$x + 1 \in E$

2. **Prove $q \to p$:** Let $x + 1 \in E$ To prove $x \in O$

$x + 1 = 2k \ ;$ k$\in Z$

$x = 2k - 1; k \in Z$ ……(1)

Since $k \in Z$, then $r = k - 1 \in Z$

$k = r + 1$……(2)

Substitute (2) in (1), $x = 2(r + 1) - 1 = 2r + 1; \quad r \in Z$

$x = 2r + 1 \in O$

Theorem1.46: x is even $\leftrightarrow$ x^2 is even

Proof: Let p: x is even number

q: x^2 is even number

من التعريف أعلاه يجب أن نبرهن بان $p \to q$ و $q \to p$

1. **Prove $p \to q$:** Let $x \in E$, $x = 2k\ ; k \in Z$

Prove $x^2 \in$ E (Theorem (1.42) مشابه لبرهان)

2. **Prove $q \to p$:** Let $x^2 \in E$ To prove $x \in E$

Take $x^2 + x = x(x + 1) \in$ E [from Theorem 1.44(ii)]

$\Longrightarrow x = x(x + 1) - x^2 \in$ E [Theorem 1.44(iii)]

$\Longrightarrow x \in E$

Theorem1.47: **(H. W.)** $\boldsymbol{x}$ is odd number if and only if $\boldsymbol{x^2}$ is odd number.

2. **Proof by Contradiction**

البرهان بالتناقض هو أن نفرض عكس المطلوب إثباته ثم نحصل على تناقض مع الفرض

Theorem1.48: Prove that: If $x^2 \in O$ then $x \in O$

Proof: Assume that $x^2 \in O$. To prove $x \in O$

By contradiction, assume that $x \in E$

$x = 2k\ ; k \in Z$

$x^2 = 4k^2 \in E$

تناقض مع الفرض لأنه في الفرض $x^2 \in O$

$\therefore x \in O.$

Theorem1.49: If x^2 is even then x is even

Proof: Assume that $x^2 \in E$. To prove $x \in E$

By contradiction, assume that $x \in O$

$x = 2k + 1\ ; k \in Z$

$x^2 = 4k^2 + 4k + 1 \in O$ تناقض مع الفرض

$\Longrightarrow x^2 \in E$. Hence, $x \in E$.

Theorem1.50: Prove that: If $n = ab$ where a and b are positive integers, then $a \leq \sqrt{n} \vee b \leq \sqrt{n}$.

Proof: Let p: $n = ab$ where a and b are positive integer **hypothesis**

q: $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ **conclusion**

The first step is to assume that the **conclusion** is false as follows:

Assume that $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is false (F). Hence, $\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$ is true (T).

$\sim(a \leq \sqrt{n} \vee b \leq \sqrt{n})$= $\sim(a \leq \sqrt{n})$ and $\sim(b \leq \sqrt{n})$ [De Morgan's law]

= $a > \sqrt{n}$ and $b > \sqrt{n}$

Multiply the two inequalities together, $ab > n$ هذه المتراجحة تناقض الفرض

This shows that $ab \neq n$ **contradiction with the hypothesis**

Thus, $a \leq \sqrt{n} \vee b \leq \sqrt{n}$ is true.

Definition1.51: Variable المتغير

An alphabetic letter $x,\ y, z, \ldots$ which represents a number that is either arbitrary or unknown.

Example1.52: ''$4x - 7 = 5$'': x is a variable

'' $\sqrt[3]{z} = 3$'': z is a variable

Definition1.53: Open Sentence الجملة المفتوحة

A sentence is called **open sentence** (or propositional function), if it contains one or more variables. Open sentence is denoted by $p(x)$, $q(x)$, $g(x)$…etc.

Example1.54: The following are open sentences:

$p(x)$: x is an odd number

$q(x, y)$: $x + y = 5$ such that $x, y \in N$

$r(z)$: $\sqrt[3]{z} = 3$ such that $z \in \mathbb{R}$

$s(y)$: computer y is working properly

Example1.55: Let the open sentence `` $\boldsymbol{p(x)}$: $x > 3$".

What are the truth value of $p(5)$ and $p(-1)$? Which values $x \in N$ that make $p(x)$ true?

Solution: $p(5)$: $5 > 3$ is a true proposition

$p(-1): -1 > 3$ is a false proposition

$p(x)$ is a true statement for $x \in \{4, 5, 6, \dots\}$.

Example1.56: Let the propositional function $\boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y})$: $x = y + 3$. What are the truth values of $q(1,2)$ and $q(3,0)$?

Solution: $q(1,2)$: $1 = 2 + 3$. This means $1 = 5$ which is false. Thus, $q(1,2)$ is false statement

$q(3,0)$: $3 = 0 + 3 = 3$. Hence $q(3,0)$ is true proposition

Example1.57: (H. W) Let the open sentence "$\boldsymbol{r}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$: $x + y = z$".

What are the truth values of $r(1,2,3)$ and $r(0,0,1)$?

Definition1.58: Solution Set (Truth set)

Let $p(x)$ be an open sentence and let A be a set. The solution set denoted by T_p is the set of all elements x of A for which $p(x)$ is true. In other words

$T_p = \{x \in A : p(x) \text{ is true}\}$

مجموعة الحل أو مجموعة الصدق: هي مجموعة العناصر التي تجعل التعبير المفتوح $p(x)$ عبارة صادقة.

Example1.59: Find the solution set for each of the following open sentences:

1) Let $p(x)$ be "$x + 2 > 7$" and $A = N$. Then

   $T_p = \{x \in N: x + 2 > 7\} = \{x \in N: x > 5\} = \{6,7, \dots\}$

   ماهي الأعداد الطبيعية الأكبر من ٥

2) Let $q(x)$ be "$x + 5 < 3$" and $A = N$. Then

   $T_q = \{x \in N: x + 5 < 3\} = \{x \in N: x < -2\} = \emptyset$

   ماهي الأعداد الطبيعية الاقل من2-.

3) Let $p(x)$ be ‘‘$x+5>1$’’ and $A=N$. Then

$$T_p=\{x\in N: x+5>1\}=\{x\in N: x>-4\}=N$$

ماهي الأعداد الطبيعية الأكبر من 4-

Example1.60: (H. W.) Find the following solution sets. Also determine $p(x)$ and A for each solution set

1) $T_P=\{x\in N: -2<x<2\}=\{1\}$

   $p(x): \ -2<x<2$ $\qquad A=N$

2) $T_P=\{x\in Z: -2<x<2\}$ **(H. W.)**

3) $T_P=\{x\in Z: -1<x<1\}$ **(H.W.)**

Example1.61: Assume we have the following statement:

$$\text{‘‘}x>2 \text{ and } x<5\text{’’}$$

Which values of $x\in N$ that make the statement true? Which values of x that make the statement false? Discuss all the possible cases.

ماهي قيم x التي تجعل العبارة أعلاه صحيحة؟ وماهي القيم التي تجعلها خاطئة؟

Solution: For $A=\{3,4\}$, we have ‘‘$x>2$ and $x<5$’’ is **true** because the values in A are greater than 2 and less than 5.

For $A^c=N-A=\{1,2,5,6,7,8,\ldots\}$, then ‘‘x > 2 and x < 5’’ is **false**

Example1.62: Assume we have the following statement:

$$\text{‘‘}x\le -3 \text{ or } x\ge 6\text{’’}$$

Which values of x $\in N$ that make the statement true? Which values of x that make the statement false?

For $A = \{x \in N: x = 6, 7, \dots\}$, the statement above is true
The statement is **false** for $A^c = N - A = \{1,2, \dots ,5\}$

Quantifiers المسورات

Quantifiers are open sentences written in a special way.

المسورات هي جمل مفتوحة مكتوبة بطريقة معينة

There are two types of quantifiers:

1. **Universal quantifiers العبارة المسورة كلياً**
2. **Existential quantifiers العبارة المسورة جزئيا**

Universal quantifiers:

Let $p(x)$ be an open sentence on a set A. The notation

‘‘$\forall\, x \in A, p(x)$’’

Denote the **universal quantification تسوير كلي** of $p(x)$ and it reads as: ‘‘for all $x, p(x)$’’ or ‘‘for every $x, p(x)$’’ or ‘‘for each $x, p(x)$’’.

The symbol ∀ is called **universal quantifier** مسوراً كلياً.

The set A is called **domain المجال**

Example1.63: $\forall\, x \in N, x > 0$

All seasons in Iraq have rain

Remark1.64: 1. The universal quantifier $p(x)$ on a domain A is **true** if and only if $T_p = A$.

2. universal quantifier $p(x)$ on a domain A is **false** if and only if there exist $x \in A$ such that $p(x)$ is false.

<u>Example</u>1.65: Find the truth value of the following open sentences:

1. $\forall\, x \in \mathbb{R},\ x + 1 > x$

Let $A = \mathbb{R}\ and\ p(x): x + 1 > x$

Because $p(x)$ is true for all $x \in \mathbb{R},$ the solution set $T_p = \mathbb{R}$

$\Longrightarrow$ the quantification $\forall\, x \in R,\ x + 1 > x$ is **true.**

2. $\forall\, x \in N, x < 2$

Let $A = N\ and\ p(x): x < 2$

$p(x)$ is not true for all $x \in N$. Take $x = 3, p(3)$ is false.

$\Longrightarrow T_p \neq N$

3. $\forall\, x \in N, (x > 0\ and\ x = 0)$

The statement is **false**, there exists $x = 4 \in N$ such that $4 > 0\ and\ 4 \neq 0.$

4. $\forall\, x \in Z, |x| > 0$ (**H. W.)**
5. For all $x \in \{1, -1\},\ \ x^2 - 1 = 0$ (**H. W.**)

<u>Existential quantifiers</u>:

Let $p(x)$ be an open sentence on a set A. The notation

‘‘$\exists\, x \in A, p(x)$’’

Denote the **existential quantification تسوير جزئي** of $p(x)$ and it read as: ‘‘there exists $x, p(x)$’’ or ‘‘there is $x, p(x)$’’ or ‘‘some $x, p(x)$’’.

The symbol $\exists$ is called **existential quantifier مسوراً جزئياً**.

The set A is called **domain المجال**

Example1.66: $\exists\ x\ \in N, x < 0$

There exist seasons in Iraq do not have rain

Remark1.67:

The existential quantifier $p(x)$ on a domain A is **true** if and only if $T_p \neq \emptyset$.

العبارة المسورة جزئياً تكون صادقة إذا وجد على الأقل عنصر واحد يحقق العبارة $p(x)$

The existential quantifier $p(x)$ on a domain A is **false** if and only if $T_p = \emptyset$.

العبارة المسورة جزئياً تكون كاذبة إذا لم يكن هناك عنصر في المجموعة A يحقق العبارة $p(x)$.

Example1.68: Find the truth value of the following open sentences:

1. $\exists x \in \mathbb{R}, x^2 = x$

A= R and p(x): $x^2 = x$

$T_p = \{0,1\}$

$\Longrightarrow$ the existential quantifier $\exists x \in \mathbb{R}, x^2 = x$ is true

2. $\exists x \in N,\ 3x + 5 = 1$

$x = \frac{-4}{3} \notin$ N

$\Longrightarrow T_p = \emptyset$

$\Longrightarrow \exists x \in N,\ 3x + 5 = 1$ is false

3. $\exists x \in Z,\ [(x + 1)^2 = 0\ and\ x^2 - 1 = 0]$

$(x + 1)^2 = 0\ \Longrightarrow x = -1$

And $\quad x^2 - 1 = 0\ \ \Longrightarrow\ \ x = -1, 1$

$T_p = \{-1\} \subset Z$

$\exists x \in Z, \ [(x+1)^2 = 0 \ and \ x^2 - 1 = 0]$ is true

<u>De Morgan's law for the existential quantifier</u>

$\sim [\exists\, x \in A, \sim p(x)] = \forall x \in A, \ p(x)$

قانون دي موركان للعلاقة بين التسوير الكلي والجزئي

<u>Example1.69</u>:

1. $\sim [\exists\, x \in E, x + 2 \notin E] = \forall x \in E, x + 2 \in E$

2. $\forall x \in N, \ \sqrt{3x} = \sqrt{3}\sqrt{x} = \sim[\exists x \in N, \sqrt{3x} \neq \sqrt{3}\sqrt{x}]$

<u>Theorem</u>1.70: Let p(x) be an open sentence and A is the domain. Then

1. $\sim[\forall x \in A, \ p(x)] = \exists\, x \in A, \sim p(x)$

2. $\sim[\forall x \in A, \ \sim p(x)] = \exists\, x \in A, \ p(x)$ (**H. W.**)

3. $\sim[\ \exists x \in A, \ p(x)] = \forall x \in A, \sim p(x)$ (**H. W.**)

<u>Proof1</u>: $\sim[\forall x \in A, \ p(x)] = \sim[\sim [\exists\, x \in A, \sim p(x)]]$ {from De Morgan}

$= \sim\sim[\exists\, x \in A, \sim p(x)]$

$= \exists\, x \in A, \sim p(x) \quad [\sim\sim p = p]$

<u>Definition1.71: </u>Nested Quantifiers المسورات المتداخلة

Two quantifiers are nested if one is within the area of the other.

في حالة وجود أكثر من متغير واحد في الجملة المفتوحة فان ذلك يستلزم وجود أكثر من مسور.

وهناك ثمانية طرق للتعبير عن المسورات المتداخلة وهي كالأتي:

Let $p(x, y)$ be an open sentence defined on the domain sets A and B. Then, the quantifiers can be expressed as follows:

1. $\forall x \in A,\ \forall y \in B,\ p(x,y)$

2. $\forall y \in B, \forall x \in A, p(x,y)$

3. $\exists x \in A,\ \exists y \in B,\ p(x,y)$

4. $\exists y \in B,\ \exists x \in A,\ p(x,y)$

5. $\forall x \in A,\ \exists y \in B,\ p(x,y)$

6. $\exists\ y \in B, \forall x \in A,\ p(x,y)$

7. $\exists\ x \in A, \forall y \in B,\ p(x,y)$

8. $\forall y \in B, \exists\ x \in A,\ p(x,y)$

Remark1.72: In the above definition, the quantifiers (1) and (2) are logically equivalent. i.e.,

$$\forall x \in A, \quad \forall y \in B, \quad p(x,y) = \forall y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (3) and (4) are logically equivalent. i.e.,

$$\exists x \in A, \quad \exists y \in B, \quad p(x,y) = \exists y \in B, \quad \exists x \in A, \quad p(x,y)$$

Example1.73:

1. $\forall x \in \mathbb{R},\ \forall y \in N, x^2 + y^2 \ \geq 0\ (\boldsymbol{True}) = \ \forall y \in N,\ \forall x \in \mathbb{R},\ x^2 + y^2 \ \geq 0\ (\boldsymbol{True})$

2. $\exists x \in N, \exists y \in N,\ \ x + 2y < 0$ **(F)** $\equiv \exists y \in N, \exists x \in N,\ \ x + 2y < 0$

(F)

Remark1.74: In the above definition, the quantifiers (5) and (6) are not logically equivalent. i.e.,

$$\forall x \in A, \quad \exists y \in B, \quad p(x,y) \neq \exists y \in B, \quad \forall x \in A, \quad p(x,y)$$

Similarly, the quantifiers (7) and (8) are not logically equivalent. i.e.,

$$\exists x \in A, \quad \forall y \in B, \quad p(x,y) \neq \forall y \in B, \quad \exists x \in A, \quad p(x,y)$$

<u>Example1.75</u>:

$\exists x \in \mathbb{R},\ \forall y \in N, x+y=0$ **(False)**

المسورة أعلاه تعني بأنه يوجد عدد حقيقي x بحيث أن حاصل جمع x و y يساوي صفر لكل عدد طبيعي y.

$\forall \mathrm{y} \in \mathrm{N},\ \exists \mathrm{x} \in \mathbb{R},\ \mathrm{x}+\mathrm{y}=0$ **(True)**

العبارة تؤكد بأنه لكل عدد طبيعي y يوجد عدد حقيقي x بحيث $\mathrm{x}+\mathrm{y}=0$.

$\Longrightarrow \exists x \in \mathbb{R},\ \forall y \in N, x+y=0 \ \neq \forall \mathrm{y} \in \mathrm{N},\ \exists \mathrm{x} \in \mathbb{R},\ \mathrm{x}+\mathrm{y}=0$.

<u>Example1.76</u>: Let $x =$ computer, $y =$ student,

$p(x,y) =$ student uses the computer

Show that $\exists x,\ \forall y, p(x,y)\ \neq \forall \mathrm{y},\ \exists \mathrm{x}, p(x,y)$

<u>Solution</u>:

$\exists x,\ \forall y,\ p(x,y)$

العبارة تعني بأنه يوجد كومبيوتر كل الطلاب تستخدمه

$\forall \mathrm{y}, \quad \exists x, \quad p(x,y)$

العبارة تعني بأنه لكل طالب يوجد كومبيوتر يستخدمه

نلاحظ أن المعنى مختلف للعبارتين المسورتين

<u>De Morgan's laws for nested quantifiers</u>

Let x *and* y are two variables defined on the sets A and B, respectively and $p(x,y)$ an open sentence. Then:

1. $\sim[\forall\ x \in A, \forall\ y \in B,\ p(x,y)] = \exists x, \exists y,\ \sim p(x,y)$ **(H. W.)**
2. $\sim[\ \exists\ x \in A, \exists\ y \in B,\ p(x,y)] =\ \forall x, \forall\ y,\ \sim p(x,y)$

3. $\sim[\forall x \in A, \exists y \in B,\ p(x,y)] = \exists x, \forall y,\ \sim p(x,y)$ **(H. W.)**
4. $\sim[\exists x \in A, \forall y \in B,\ p(x,y)] = \forall x, \exists y,\ \sim p(x,y)$ **(H. W.)**

Proof 2:

Take the L. H. S

$\sim[\exists x \in A, \exists y \in B,\ p(x,y)] = \forall x \in A \sim [\exists y \in B,\ p(x,y)]$

$= \forall x \in A, \forall y \in B,\ \sim p(x,y)$

= R. H. S

Example1.77: Find the truth values of the following statements and of their negations:

1. $\forall x \in \mathbb{R}\ (x \neq 0), \exists y \in \mathbb{R},\ xy = 1$

The statement is **true** because $\forall x \in \mathbb{R}\ (x \neq 0),\ \exists y = \frac{1}{x} \in \mathbb{R},\ x\frac{1}{x} = 1$

Negation:

$\sim[\forall x \in \mathbb{R}\ (x \neq 0), \exists y \in \mathbb{R},\qquad xy = 1]$

$= \exists x \in \mathbb{R}\ (x \neq 0),\ \forall y \in \mathbb{R},\ xy \neq 1$

The statement is **false**

Let $x = 2$ and $y = \frac{1}{2}$ then $xy = 1$

2. $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0$ is true

يوجد عددين حقيقيين حاصل جمع مربعيهما عدد غير سالب

Negation:

$\sim[\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x^2 + y^2 \geq 0]$

$= \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 < 0$ is **false**

3. $\forall x \in N, \forall y \in N, x + y \in N$ **(H. W.)**

4. $\forall x \in N, \exists y \in Z, x + y \in N$ **(H. W.)**

Exercise1.78:

1. Express the following using connective operators and/or quantifiers

عبر عما يلي باستخدام ادوات الربط او المسورات

i) there exists p, and there exist q such that $pq = 32$

ii) for each x, there exists y such that $x < y$

iii) each even number is not odd number

iv) for each x, if x is natural number then x is an integer number

v) for each natural number x, x is even number or x is odd number

2. Find the negation of the following sentences:

i) $\forall x, \forall y, \exists z, x + y + z = 18$

ii) there exists y such for each $x, xy \leq 2$

iii) $\exists x, [p(x) \rightarrow q(x)]$

Foundation Of Mathematics

أسس الرياضيات

الأستاذ: عبدالله محمود جاسم

CHAPTER TWO

Set Theory

قسم الرياضيات

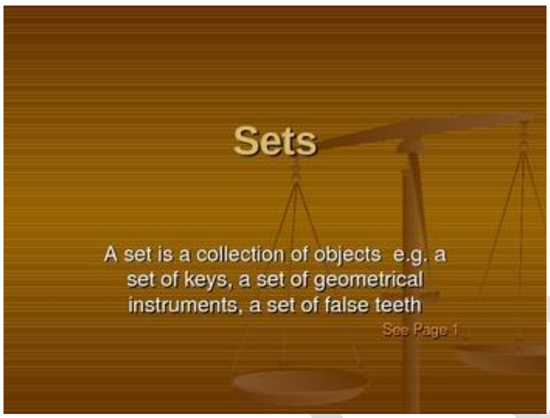
المرحلة : الاولى

السنة الدراسية:٢٠٢٥-٢٠٢٦

CHAPTER TWO

Set Theory

نظرية المجموعات



Chapter Two Contents:

1. Basic notion of sets مفهوم المجموعات

2. Subsetsالمجموعات الجزئية

3. Algebra of sets (union, intersection, difference, complement, symmetric difference) جبر المجموعات أو العمليات على المجموعات

Definition 2.1: Set

A set is an unordered collection of objects. The objects are called **the elements** or **members** of the set.

المجموعة هي تجمع من الأشياء المعرفة بدون ترتيب والتي تسمى بالعناصر أو أعضاء تنتمي للمجموعة.

Remark 2.2

1. The capital letters usually used to represent sets such as $A, B, C,$…etc.
2. The small letters such as $a, b, c, d,$…etc are used to represents the members or the elements of the set.

3. Membership in a set is denoted as follows: انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$\boldsymbol{a} \in \boldsymbol{A}$ denotes that $\boldsymbol{a}$ **belongs to** a set A

4. Non-membership to a set is denoted as follows: عدم انتماء عنصر لمجموعة يعبر عنه بالشكل التالي

$\boldsymbol{a} \notin \boldsymbol{A}$ denotes that $\boldsymbol{a}$ **does not belong to** a set A

Specifying a Set: طرق التعبير عن المجموعة

1. Listing members of a set: الطريقة الجدولية

In this way, we list all non-repeated members of a set separated by commas and contained in braces { }. The members are not in an order.

الطريقة الجدولية أو طريقة القائمة: في هذه الطريقة نضع جميع العناصر الغير المعادة بين قوسي مجموعة وبفواصل تفصل بينها. عناصر المجموعة لايشترط أن تكون مرتبة بطريقة معينة

Example2.3:

1. $A = \{1,2,-5,0,9\}$, $B = \{x, y, \text{Ali}, \text{fish}\}$, $C = \{y_1, y_2, y_3\}$ are sets

2. The set of vowel letters in English: $V = \{a, e, i, o, u\}$

3. The set of even positive numbers less than 8 is: $W = \{0, 2, 4, 6\}$.

4. The set of positive numbers less than 50 is: $K = \{1, 2, \ldots, 49\}$

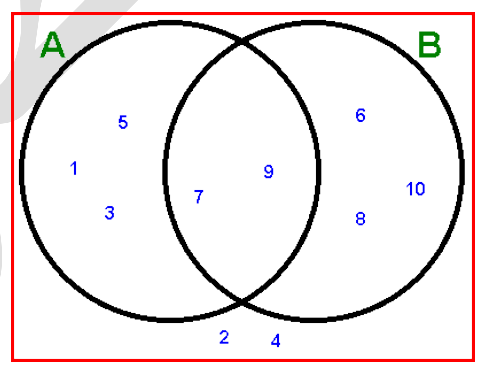
2. Listing a set property: استخدام الصفة المميزة للمجموعة

In this way, we state the property that characterize the elements in a set in as follows: $\{x: p(x)\}$, where x is a variable and $p(x)$ is an open sentence.

Example2.4: $A = \{x: x \in Q\}$
$B = \{x: x \text{ is positive odd and } x < 10\} = \{1,3,5,7,9\}$
$C = \{x \in N: -3 \le x \le 5\} = \{1,2,3,4,5\}$

3. Venn Diagrams: مخططات فن

في هذه الطريقة توضع عناصر المجموعة داخل منحني مغلق يمثل المجموعة وتستخدم هذه الطريقة لأغراض توضيحية فقط.



Definition2.5: Empty Set المجموعة الخالية

The set that contains no elements is called an empty set and is denoted by $\{\ \}$ or $\emptyset$.

تسمى المجموعة التي لاتحوي أي عنصر بالمجموعة الخالية.

Example2.6: $A = \{x \in N: 2 < x < 3\} = \emptyset$

$B = \{x \in E: \sqrt{x}=1\} = \emptyset$

$C = \{x \in N: x < 0\} = \{\ \}$

Subsets: المجموعات الجزئية
The set A is a subset of a set B ($A \subseteq B$) if and only if every element of A is an element of B. In other words,

$$A \subseteq B \text{ iff } \forall x,\ x \in A \Rightarrow x \in B$$

Remark2.7: A is **not a subset** of B is denoted by $A \nsubseteq B$.

$$A \nsubseteq B \text{ if and only if } \sim[\forall x, x \in A \Rightarrow x \in B]$$
$$\text{if and only if } \exists x; x \in A \wedge x \notin B$$

Example2.8: Consider the sets A= {2}, B= {1, 2, 3} and C = {4, 5} and D= {-2, 1, 2, 3, 4, 5}. Then $A \subseteq B, . A \subseteq D,\ B \subseteq D \text{ and } C \subseteq D.$
It is true that $A \subseteq A,\ B \subseteq B,\ C \subseteq C \text{ and } D \subseteq D$.

Example2.9: Let A= {4, 9} and $B = \{x \in N: 1 < x < 10\}$. Determine whether $A \subseteq B$ or $B \subseteq A$.

Solution: The set B be can be written as B={2,…,9}. Then
$$\forall x,\ x \in A \Rightarrow x \in B$$
Hence, $A \subseteq B$.
But $B \nsubseteq A$ because, for example, $\exists x = 5 \in B \wedge x \notin A$.

Example2.10: Let A= $\{x \in N: x > 3\}$ and $B = \{x \in N: x^2 > 4\}$. Is $A \subseteq B$? Is $B \subseteq A$?

Solution: Let $x \in A \Rightarrow\ x \in N\ and\ x > 3$

$\Rightarrow\ x^2 > 9$

$\Rightarrow\ x^2 > 4 \Rightarrow x \in B.$

$\Rightarrow A \subseteq B.$

Is $B \subseteq A$?

Example2.11: (H. W.) Let $A = \{-2, 3\}, B = \{x \in Z: x^3 - x^2 - 6x = 0\}$. Determine whether $A \subseteq B$ or $B \subseteq A$?

Example2.12: (H. W.) Let $A = \{x \in N : x \geq 4\}$ and $B = \{x \in N : x < 9\}$. Determine whether $A \subseteq B$ or $B \subseteq A$?

Theorem2.13: Let A, B and C be any sets, then

1. $\emptyset \subseteq A$
2. $A \subseteq A$
3. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

Proof 1: T. P $\emptyset \subseteq A$, *i.e.*, T. P $\forall x \in \emptyset \Rightarrow x \in A$

F ⇒ (T or F) = T

$\Rightarrow\ \emptyset \subseteq A.$

Proof 2: T. P $A \subseteq A$, *i.e.*, T. P $\forall x \in A \Rightarrow x \in A$

$T \Rightarrow T = T$

$\Rightarrow\ A \subseteq A.$

Proof 3: T. P If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

T. P $\forall x, x \in A \Rightarrow x \in C$

$\because A \subseteq B \Rightarrow \forall x, x \in A \Rightarrow x \in B$

$\because B \subseteq C \Rightarrow \forall x, x \in B \Rightarrow x \in C$

$\therefore \forall x, x \in A \Rightarrow x \in B \Rightarrow x \in C$

$\therefore \forall x, x \in A \Rightarrow x \in C$

$\therefore A \subseteq C.$

<u>Definition</u>2.14: Proper Subset المجموعة الجزئية الفعلية

A set A is called a **proper** subset of B and denoted by $(A \subset B)$ if and only if $A \subseteq B$ and there exist an element x ∈ B that is x $\notin A$.

i.e., $A \subset B$ iff $\{\forall x, x \in A \Rightarrow x \in B\} \wedge \{\exists y, y \in B \wedge y \notin A\}$

<u>Example2.15</u>: Let $A = \{x \in N \text{ v } x^2 - 16 = 0\}$
$B = \{x \in N\colon\ x^2 - 16 = 0\}$
Determine if $A \subset B$ or $B \subset A$.

<u>Solution</u>: $A = \{1, 2, 3, \dots\} \cup \{4, -4\}$ and $B = \{4\}$
It is clear that $B \subset A$ because $B \subseteq A$ and
$\exists y = \{1,2,3,5, \dots\} \in A \wedge y \notin B$.

<u>Example2.16</u>: (H. W.) Let $A =$ {fish, dog, bird}, $B = \{x, y, z, w\}$.
Determine if $A \subset B$ or $B \subset A$.

<u>Example2.17</u>: (H. W.) Let $A = \{x \in Z\colon -2 \le x \le 10\}$
$B = \{x \in Z \text{ v } x^2 + 9 = 0\}$
Determine if $A \subset B$ or $B \subset A$.

<u>Solution</u>: $A = \{-2, -1, 0, 1, \dots, 10\}$ and $B = \{0, \pm 1, \pm 2, \dots\} \cup \{3i, -3i\}$

Complete the solution!

<u>Definition2.18:</u> Equal Sets المجموعات المتساوية

Two sets A and B are equal if they both have the same elements or, equivalently, if each is contained in the other.
يقال أن المجموعة A تساوي المجموعة B إذا كانت لهما نفس العناصر أو إذا كانت كل مجموعة جزئية من الأخرى.

$$A = B \ iff \ A \subseteq B \ \wedge \mathrm{B} \subseteq A$$
$$\leftrightarrow \{ \forall x, x \in A \rightarrow x \in B\} \ \wedge \ \{ \forall x, x \in B \rightarrow x \in A\}$$
$$\leftrightarrow \{ \forall x, x \in A \leftrightarrow x \in B\}$$

<u>Example2.19</u>: Let $A = \{x \in Z \ \wedge 5\mathrm{x}^2 + 2 = 0\}$
$B = \{x \in N: 2x + 3 = 0\}$

Is $A = B$?

<u>Solution</u>: $A = Z \ \cap \{\mp\sqrt{\frac{2}{5}}\ i\} = \emptyset$

$B = \emptyset$

$\Longrightarrow A = B$

<u>Lemma2.20:</u> (**H. W.**) Prove that: $A = A$, for any set A.

<u>Definition2.21</u>: Universal Set المجموعة الشاملة

Universal set U is the set that contains all the elements or the sets we have under discussion.

المجموعة الشاملة: هي المجموعة التي تحوي جميع العناصر أو المجموعات قيد المناقشة ويرمز لها بالرمز U.

<u>Example2.22</u>: Let $A = \{x, y, 3\}, B = \{2, -5,100\}, C = \{2, 3, 1\}$
Find a universal set U.

<u>Example2.23</u>: Let $A = \{x \in R: 2 \leq x \leq 5\} \ and \ \ B = \{x \in R: \ -1 \leq x \leq 2\}$
Find a universal set U.

<u>Definition2.24:</u> Family of Sets عائلة المجموعات

Family of sets is a set that have other sets as members.

يقال للمجموعة التي يكون كل عنصر من عناصرها مجموعة أنها **عائلة مجموعات**

Example2.25:

1. $A = \{\{1\}, \{2\}\}$ is a family of sets

2. $B = \{\emptyset\}$ is a family of a set

3. $X = \{X\}$ is a family of a set

4. $A = \{x, \{y, z\}, \{1, \dots, 5\}\}$

5. $H = \{\, A: A \text{ is a subset of } \{1,2,3\}\}$

6. $K = \{A_i: A_i = \{2, \frac{2}{i}\}, i = 1, 2, 3\}$

Definition2.26: Power Set مجموعة القوى أو مجموعة الأجزاء

Given a set X, the **power set of X** is the set of all subsets of X. The power set of X is denoted by $P(X)$.

لتكن X مجموعة يقال لمجموعة كل المجموعات الجزئية من X أنها مجموعة القوى ل X ويرمز لها بالرمز $P(X)$.

$$\boxed{P(X) = \{A: A \subseteq X\}, \qquad A \in P(X) \Leftrightarrow A}$$

Example2.27: Find $P(X)$ for the following sets X:

1. X=$\{1, 2, a\}$, $\quad P(X) = \{\emptyset, X, \{1\}, \{2\}, \{a\}, \{1,2\}, \{1, a\}, \{2, a\}\}$

2. $X = \{\emptyset\}$, $\qquad P(X) = \{\emptyset, X\}$

3. $X = \{\{-2\}, 3\}$, $\quad P(X) = \{\emptyset, X, \{\{-2\}\}, \{3\}\}$

Remark2.28: 1. Since $X \subseteq X, then\ P(X) \neq \emptyset$.

2. If X is finite and has n elements, then $P(X)$ has 2^n elements.

Theorem2.29: Let X, Y be any sets, then $X \subseteq Y \iff P(X) \subseteq P(Y)$.

Proof: ($\Longrightarrow$) Let $X \subseteq Y$ T.P $P(X) \subseteq P(Y)$

Let $A \in P(X) \Longrightarrow A \subseteq X$ (By def. of $P(X)$)

$\Longrightarrow A \subseteq Y \quad (X \subseteq Y)$

$\Longrightarrow A \in P(Y)$

$\therefore P(X) \subseteq P(Y)$

($\Longleftarrow$) Let $P(X) \subseteq P(Y)$ To Prove $X \subseteq Y$

Let $x \in X \Longrightarrow \{x\} \subseteq X$

$\Longrightarrow \{x\} \in P(X)$

$\Longrightarrow \{x\} \in P(Y) \ \{ P(X) \subseteq P(Y)\}$

$\Longrightarrow \{x\} \subseteq Y$

$\Longrightarrow x \in Y$

$\therefore X \subseteq Y$

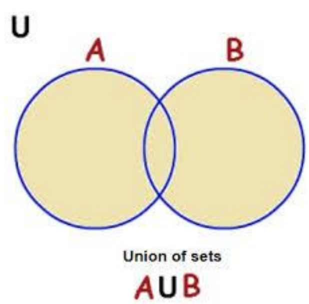
Algebra of sets:

1. Union الاتحاد

The **union** of the sets A and B, denoted by $\boldsymbol{A} \cup \boldsymbol{B}$, is the set of elements which belong to A **or** to B.

إتحاد مجموعتين B و A هي مجموعة العناصر التي تنتمي للمجموعة A أو المجموعة B.

$$A \cup B = \{x, x \in A \vee x \in B\}$$
$$x \in A \cup B \iff x \in A \vee x \in B$$
$$x \notin A \cup B \iff x \notin A \wedge x \notin B$$



Example2.30: Let $A = \{x \in N: 1 \leq x \leq 5\} = \{1,2,3,4,5\}$

$B = \{x \in N: 8 \leq x \leq 12\} = \{8,9,10,11,12\}$

Find $A \cup B,\ B \cup A, A \cup A,\ and\ B \cup \emptyset$

Solution: $A \cup B = B \cup A = \{1, \dots, 5, 8, \dots, 12\}$

$A \cup A = A$

$B \cup \emptyset = B$

<u>Example2.31</u>: Let $A = \{x \in R : -2 \le x \le 5\} = [-2,5]$,
$B = \{x \in E : x^2 - 16 = 0\} = \{4, -4\}$
$C = \{1, 4\}$
Find $A \cup (B \cup C), (A \cup B) \cup C, P(B), P(C), P(B \cup C)$

<u>Definition</u>2.32: Generalization of the union تعميم الاتحاد
Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

اتحاد عدد منتهي من المجموعات

In general,

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup \dots \cup A_n \cup A_{n+1} \cup \dots$$

اتحاد عدد غير منتهي من المجموعات

<u>Example2.33</u>: Let $H = \{A_i;\ A_i = \{2i + 3\},\ i \in Z\}$
Find

$\bigcup_{i=1}^{4} A_i$, and $\bigcup_{i=-1}^{3} A_i$

<u>Solution</u>: $\bigcup_{i=1}^{4} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 = \{5\} \cup \{7\} \cup \{9\} \cup \{11\}$
$= \{5,7,9,11\}$

$\bigcup_{i=-1}^{3} A_i = A_{-1} \cup A_0 \cup A_1 \cup A_2 \cup A_3 = \{1\} \cup \{3\} \cup \{5\} \cup \{7\} \cup \{9\}$
$= \{1,3,5,7,9\}$

<u>Example2.34</u>: Let $K = \{A_n;\ A_n = (-n, n),\ n \in N\}$
Find $A_1 \cup \bigcup_{n=2}^{\infty} A_n$

<u>Solution</u>: $A_1 \cup \bigcup_{n=2}^{\infty} A_n = (-1,1) \cup (-2,2) \cup \dots \cup (-k, k) \cup \dots$
<u>Example2.35</u>: (H. W.) Let $K = \{A_n;\ A_n = [n, n + 10),\ n \in Z\}$
Find $\bigcup_{n=-2}^{2} A_n$

Example2.36: **(H. W.)** Let $K=\{A_j; A_j=\{j+1, j+2\},\ j\in N\}$
Find $\bigcup_{j=3}^{10} A_j$

Theorem2.37: Let $A, B\ and\ C$ any three sets. Then:
1. $A\cup\emptyset=A$ (Identity law)
2. $A\cup A=A$ (Idempotent law)
3. $A\cup U=U$ (Domination law)
4. $(A\cup B)\cup C=A\cup(B\cup C)$ **(H.W.)**
5. $A\cup C=C\cup A$
6. $A\subseteq B \Leftrightarrow A\cup B=B$
   $B\subseteq A \Leftrightarrow A\cup B=A$
7. $A\subseteq A\cup B$ **(H.W.)**
   $B\subseteq A\cup B$
8. $P(A)\cup P(B)\subseteq P(A\cup B)$

Proof 1: To prove $A\cup\emptyset\subseteq A\wedge A\subseteq A\cup\emptyset$

T. P $A\cup\emptyset\subseteq A$ (T. P $\forall x\in A\cup\emptyset\Longrightarrow x\in A$)
Let $x\in A\cup\emptyset\Longrightarrow x\in A\vee x\in\emptyset$ (def. of $\cup$)
$\Longrightarrow x\in A\vee$ F
$\Longrightarrow x\in A$ (p v F=p)
$\therefore A\cup\emptyset\subseteq A$ ……………**(1)**

Let $x\in A\Longrightarrow x\in A\vee$ F (p $\vee$ F=p)
$\Longrightarrow x\in A\vee x\in\emptyset$
$\Longrightarrow x\in A\cup\emptyset$ (def. of $\cup$)
$\therefore A\subseteq A\cup\emptyset$ ……………**(2)**

From (1) & (2), $A\cup\emptyset=A$

Proof 2: To prove $A\cup A\subseteq A\wedge A\subseteq A\cup A$

T. P $A\cup A\subseteq A$ (T. P $\forall x\in A\cup A\Longrightarrow x\in A$)
Let $x\in A\cup A\Longrightarrow x\in A\vee x\in A$ (def. of $\cup$)
$\Longrightarrow x\in A$ $(p\vee p=p)$
$\therefore A\cup A\subseteq A$ ……………**(1)**

Let $x\in A\Longrightarrow x\in A\vee x\in A$ $(p\vee p=p)$

$\Longrightarrow x \in A \cup A$ (def. of $\cup$)
$\therefore A \subseteq A \cup A$ **……………(2)**

From (1) & (2), $A \cup A = A$
<u>Proof 3</u>: To prove $A \cup U \subseteq U \wedge U \subseteq A \cup U$

T. P $A \cup U \subseteq U$ (T. P $\forall x \in A \cup U \Longrightarrow x \in U$)
Let $x \in A \cup U \Longrightarrow x \in A \vee x \in U$ (def. of $\cup$)
$\Longrightarrow x \in U \vee x \in U$ ($A \subseteq U$)
$\Longrightarrow x \in U$
$\therefore A \cup U \subseteq U$ **……………(1)**

Let $x \in U \Longrightarrow x \in U \vee x \in A$ (T $\vee$ p=T)
$\Longrightarrow x \in A \cup U$ (def. of $\cup$)
$\therefore U \subseteq A \cup U$ **……………(2)**
From (1) & (2), $A \cup U = U$
<u>Proof 5</u>: To prove $A \cup C \subseteq C \cup A \wedge C \cup A \subseteq A \cup C$

T. P $A \cup C \subseteq C \cup A$
Let $x \in A \cup C \Longrightarrow x \in A \vee x \in C$ (def. of $\cup$)
$\Longrightarrow x \in C \vee x \in A$ ($\vee$ is commutative)
$x \in C \cup A$ (def. of $\cup$)

$\therefore A \cup C \subseteq C \cup A$ **……………(1)**

Similarly, show that $C \cup A \subseteq A \cup C$ (**H. W.**) **……………(2)**

From (1) & (2), $A \cup C = C \cup A$

<u>Proof 6</u>: To prove $A \subseteq B \Leftrightarrow A \cup B = B$

($\Longrightarrow$) if $A \subseteq B$ T. P $A \cup B = B$
T. P $A \cup B \subseteq B \wedge B \subseteq A \cup B$
Let $x \in A \cup B \Longrightarrow x \in A \vee x \in B$ (def. of $\cup$)
$\Longrightarrow x \in B \vee x \in B$ (by hypo. $A \subseteq B$)
$\Longrightarrow x \in B$ ($p \vee p = p$)
$\therefore A \cup B \subseteq B$ **……………(1)**

Let $x \in B \Longrightarrow x \in B \ \lor \ x \in A$ (T $\lor\, p$ =T)
$\Longrightarrow x \in \ A \cup B$ (def. of $\cup$)

$$\therefore B \subseteq A \cup B \ \text{...............}\mathbf{(2)}$$

From (1) & (2), $A \cup B = B$

($\Longleftarrow$) If $A \cup B = B$ To prove $A \subseteq B$
Let $x \in A \Longrightarrow x \in A \ \lor \ x \in B$ (T $\lor\, p$ =T)
$\Longrightarrow \quad x \in \ A \cup B$ (def. of $\cup$)
$\Longrightarrow \quad x \in \ B$ (by hypo. $A \cup B = B$)

$\therefore A \subseteq B$

<u>Proof 8</u>: T. P $P(A) \cup P(B) \subseteq P(A \cup B)$

Let $X \in \ P(A) \cup P(B)$ To prove $X \in P(A \cup B)$

$X \in \ P(A) \cup P(B) \Longrightarrow X \in P(A) \ \lor \ X \in P(B)$ (def. of $\cup$)
$\Longrightarrow X \subseteq A \ \lor \ X \subseteq B$ (def. of $P(A)$)
$\Longrightarrow X \subseteq A \cup B$ (def. of $\cup$)
$\Longrightarrow X \in P(A \cup B)$ (def. of $P(A \cup B)$)

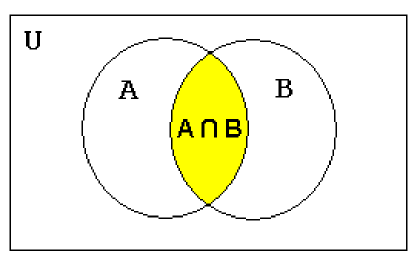
$$\therefore \ P(A) \cup P(B) \subseteq P(A \cup B)$$

2. <u>Intersection</u> التقاطع

The **intersection** of the sets A and B, denoted by $\boldsymbol{A \cap B}$, is the set of elements which belong to both A **and** to B.

تقاطع مجموعتين B و A هي مجموعة العناصر التي تنتمي لكلا المجموعتين معاً.

$A \cap B = \{x; x \in A \ \land \mathrm{x} \in \mathrm{B}\}$
$x \in \ A \cap B \Leftrightarrow x \in A \ \land \mathrm{x} \in \mathrm{B}$
$x \notin \ A \cap B \Leftrightarrow x \notin A \ \lor \ \mathrm{x} \notin \mathrm{B}$



Example2.38: Let $A = [2,9], B = (5,14], C = (8,12)$
Find $A \cap B, A \cap \emptyset, B \cap B, B \cap C, A \cap C, (A \cap B) \cup (B \cap C)$

Definition2.39: Generalization of the intersection تعميم التقاطع
Let $A_1, A_2, \dots, A_n, A_{n+1}, \dots$ be any sets. Then:

$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i = \{x, x \in A_i \ \ \forall i = 1,2,\dots,n\}$
تقاطع عدد منتهي من المجموعات

In general,

$A_1 \cap A_2 \cap \dots \cap A_n \cap A_{n+1} \cap \dots = \bigcap_{i=1}^{\infty} A_i = \{x, x \in A_i, \ \forall\, i\}$
تقاطع عدد غير منتهي من المجموعات

Example2.40: Let $X = \{A_i; A_i = \{1,2,3,\dots,i\}; i \in N\}$
Find

$$\left(\bigcup_{i=1}^{5} A_i\right) \cap \left(\bigcap_{i=1}^{5} A_i\right)$$

Solution: $A_1 = \{1\}, A_2 = \{1,2\}, \dots \ A_5 = \{1,2,3,4,5\}$

$$\bigcup_{i=1}^{5} A_i = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = \{1,2,3,4,5\}$$

$$\bigcap_{i=1}^{5} A_i = A_1 \cap \dots \cap A_5 = \{1\}$$

$\left(\bigcup_{i=1}^{5} A_i\right) \cap \left(\bigcap_{i=1}^{5} A_i\right) = \{1\}$

Example2.41: Let $X = \left\{A_j; A_j = \left(\frac{-1}{j}, \frac{1}{j}\right); j \in N\right\}$
Find

$$\bigcap_{j=2}^{4} A_j \quad and \quad \bigcup_{j=2}^{4} A_j$$

Solution:

$$\bigcap_{j=2}^{4} A_j = A_2 \cap A_3 \cap A_4 = \left(-\frac{1}{2},\frac{1}{2}\right) \cap \left(-\frac{1}{3},\frac{1}{3}\right) \cap \left(-\frac{1}{4},\frac{1}{4}\right) = \left(-\frac{1}{4},\frac{1}{4}\right)$$

$$\bigcup_{j=2}^{4} A_j = A_2 \cup A_3 \cup A_4 = \left(-\frac{1}{2},\frac{1}{2}\right) \cup \left(-\frac{1}{3},\frac{1}{3}\right) \cup \left(-\frac{1}{4},\frac{1}{4}\right) = \left(-\frac{1}{2},\frac{1}{2}\right)$$

Example2.42: (H. W.) Let $X = \{A_n; A_n = \{n^3 + 1\}; n \in Z\}$
Find

$\bigcap_{n=-3}^{0} A_n$, $\bigcap_{n=1}^{\infty} A_n$, $P(\bigcup_{n=1}^{3} A_n)$

Example2.43: (H. W.) Let $X = \{A_n;\ A_n = \{n-2, n-1, n\}; n \in N\}$
Find

$\bigcap_{n=1}^{\infty} A_n, \bigcup_{n=1}^{\infty} A_n$

Theorem2.44: Let $A, B \text{ and } C$ any three sets. Then:

1. $A \cap \emptyset = \emptyset$ (**H. W.**) (Domination law)
2. $A \cap A = A$ (**H. W.**) (Idempotent law)
3. $A \cap U = A$ (Identity law)
4. $(A \cap B) \cap C = A \cap (B \cap C)$
5. $A \cap C = C \cap A$ (**H. W.**)
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (**H. W.**)
8. $A \cap B \subseteq A$ (**H. W.**)
   $A \cap B \subseteq B$
9. $A \subseteq B \iff A \cap B = A$
10. $P(A) \cap P(B) = P(A \cap B)$ (**H. W.**)

Proof 3: **T. P** $A \cap U \subseteq A \wedge A \subseteq A \cap U$

Assume that $x \in A \cap U \Longrightarrow \mathrm{x} \in A \wedge \mathrm{x} \in U$

$\Longrightarrow \mathrm{x} \in A$

$\therefore A \cap U \subseteq A$ **…………..(1)**

Let $x \in A$ T.P $x \in A \cap U$

$x \in A \Longrightarrow x \in A \wedge \mathrm{x} \in U \quad [A \subseteq U]$

$\Longrightarrow x \in A \cap U$ **…………..(2)**

From (1) & (2), $A \cap U = A$

Proof 4:

Let $x \in (A \cap B) \cap C \Leftrightarrow x \in (A \cap B) \wedge x \in C$ (def. of ∩)

$\Leftrightarrow (x \in A \wedge x \in B) \wedge x \in C$ (def. of ∩)

$\Leftrightarrow x \in A \wedge (x \in B \wedge x \in C)$ (∧ is assoc.)

$\Leftrightarrow x \in A \wedge x \in B \cap C$ (def. of ∩)

$\Leftrightarrow x \in A \cap (B \cap C)$ (def. of ∩)

$\therefore (A \cap B) \cap C = A \cap (B \cap C)$

Proof 6:

Let $x \in A \cap (B \cup C) \Leftrightarrow x \in A \wedge x \in B \cup C$ (def. of ∩)

$\Leftrightarrow x \in A \wedge (x \in B \vee x \in C)$ (def. of ∪)

$\Leftrightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$ (distribute ∧ on v)

$\Leftrightarrow x \in A \cap B \vee x \in A \cap C$ (def. of ∩)

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ (def. of ∪)

Proof 9: **T. P** $A \subseteq B \Leftrightarrow A \cap B = A$

($\Longrightarrow$) Suppose $A \subseteq B$ T.P $A \cap B \subseteq A \wedge A \subseteq A \cap B$

Let $x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of ∩)

$\Longrightarrow x \in A$

$\therefore A \cap B \subseteq A$**………….(1)**

Let $x \in A \Longrightarrow x \in A \wedge x \in B \quad [A \subseteq U]$

$\Longrightarrow x \in A \cap B$

$\Longrightarrow A \subseteq A \cap B$**.......(2)**

From (1) & (2), $A \cap B = A$

($\Longleftarrow$) Let $A \cap B = A$ **T. P** $A \subseteq B$ (**H. W.**)

3. <u>The Complement</u> المتممة أو المكملة

Let U be a universal set and A be any subset of U. The **complement** of a set A, denoted by A^c, is the set of elements which belong to U but do not belong to A.

متممة المجموعة A هي مجموعة العناصر التي تنتمي للمجموعة الشاملة ولا تنتمي للمجموعة A.

$$\boxed{\begin{array}{l} A^c = \{x, x \in U \ \wedge \ x \notin A\} = U \backslash A \\ x \in A^c \Leftrightarrow x \notin A \\ x \in \mathrm{A} \Leftrightarrow \mathrm{x} \notin \mathrm{A^c} \end{array}}$$

<u>Example2.45</u>: Let $U = Z, A = \{-1, 0, 1\}$. Find A^c.

<u>Solution</u>: $A^c = Z \backslash A$

<u>Example2.46</u>: Let $U = [0, 8), A = \{1, 2\}$. Find A^c.

$A^c = [0,1) \cup (1,2) \cup (2,8)$

<u>Example2.47</u>: Let $U = \{1, 2, \dots, 10\}$,

$A = \{x \in N: 1 \leq x \leq 3\} = \{1,2,3\}$

$B = \{x \in N: 8 \leq x \leq 10\} = \{8,9,10\}$

$C = \{x \in N: 1 \leq x \leq 2\} = \{1,2\}$

Find $A^c, B^c, C^c, (A \cup B)^c, (A \cap C)^c, (C \cup B)^c$

<u>Solution</u>: $A^c = \{4,5, \dots ,10\}$

$B^c = \{1,2, \dots ,7\}$

$C^c =$

$(A \cup B)^c = \{1,2,3,8,9,10\}^c = \{4,5,6,7\}$

$(A \cap C)^c = \{1,2\}^c = \{3,4, \dots ,10\}$

<u>Theorem2.</u>48: Let $A\ and\ B$ any two sets. Then:

1. $\emptyset^c = U,\ U^c = \emptyset$
2. $A^c \cap A = \emptyset$ (**H. W.**), $A^c \cup A = U,$ $(A^c)^c = A$ (**H. W.**)
3. $(A \cup B)^c = A^c \cap B^c$
4. $(A \cap B)^c = A^c \cup B^c$ (**H. W.**)
5. $A \subseteq B \Leftrightarrow B^c \subseteq A^c$ (**H. W.**)
6. $A \subseteq B \Leftrightarrow A \cap B^c = \emptyset$

<u>Proof 1</u>: T. P $\emptyset^c = U$
Assume that $\emptyset^c \neq U$ برهان غير مباشر
$\exists x, x \in \emptyset^c \wedge x \notin$ U
$x \in \emptyset^c \Longrightarrow x \in U \wedge x \notin \emptyset$ (def. of $\emptyset^c$)
$\Longrightarrow x \in U$ تناقض مع الفرض
$\therefore\ \emptyset^c = U$

<u>Proof 1</u>: T. P $U^c = \emptyset$
Assume that $U^c \neq \emptyset$ برهان غير مباشر
$\exists x, x \in U^c \wedge x \notin \emptyset$
$x \in U^c \Longrightarrow x \in U \wedge x \notin$ U تناقض
$\therefore U^c = \emptyset$

<u>Proof 2</u>: T. P $A^c \cup A = U$
Assume that $A^c \cup A \neq U$ برهان غير مباشر
$\exists x, x \in A^c \cup A \wedge x \notin$ U ... ... $(*)$
$x \in A^c \cup A \Longrightarrow x \in A^c \vee x \in A$ (def. of $\cup$)
If $x \in A^c \Longrightarrow \boldsymbol{x} \in \boldsymbol{U} \wedge x \notin$ A (contradiction with $x \notin$ U in *)
If $x \in A \Longrightarrow \boldsymbol{x} \in \boldsymbol{U} \wedge x \notin A^c$ (contradiction with $x \notin$ U in *)
$\therefore A^c \cup A = U$

<u>Proof 3</u>: T. P $(A \cup B)^c = A^c \cap B^c$
$x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$ (def of complement)
$\Leftrightarrow x \notin A \wedge x \notin B$ (def of $\cup$)
$\Leftrightarrow x \in A^c \wedge x \in B^c$ (def. of A^c)
$\Leftrightarrow x \in A^c \cap B^c$ (def. of $\cap$)

<u>Proof 6</u>: ($\Longrightarrow$) Let $A \subseteq B$ **T. P** $A \cap B^c = \emptyset$
Assume that $A \cap B^c \neq \emptyset$ برهان غير مباشر بالتناقض
$\exists x, x \in A \cap B^c \Longrightarrow x \in A \wedge x \in B^c$

$\Longrightarrow x \in B \ \wedge \ x \in B^c$ (by hypo. $A \subseteq B$)
$\Longrightarrow$ تناقض
$\therefore A \cap B^c = \emptyset$

($\Longleftarrow$) Let $A \cap B^c = \emptyset$ T. P $A \subseteq B$
Let $x \in A \Longrightarrow x \notin B^c$ (by hypo. $A \cap B^c = \emptyset$)
$\Longrightarrow x \in B$ (def. of B^c)
$\therefore \ A \subseteq B$

4. Difference or relative complement : الفضلة أو الفرق

Let A and B are two sets. The **difference** between A and B, denoted as A-B or $A\backslash B$, is the set of elements which belong to A but do not belong to B.

يقال لمجموعة العناصر المنتمية إلى Aوغير منتمية إلى B بأنها فضلة Aعلى B.

$$A - B = \{x, x \in A \ \wedge \ x \notin B\} = A\backslash B$$
$$x \in A - B \Leftrightarrow x \in A \ \wedge \ x \notin B$$
$$x \notin A - B \Leftrightarrow x \notin A \ \vee \ x \in B$$

Example2.49: Let $A = \{x \in Z: x \geq 1\} = N$

$$B = \{x \in R: x^2 + 3 = 0\} = \emptyset$$

$$C = \{x \in Z: -3 < x \leq 4\} = \{-2, -1, \dots, 4\}$$

$$D = \{x \in O: x^2 - 9 = 0\} = \{3, -3\}$$

Find $A\backslash D$, $B - B$, $C\backslash(A \cap D)$, $(B \cup A)\backslash C$, $(C \cup A)\backslash(B \cap D)$

Solution: $A\backslash D = N\backslash\{\mathbf{3}, -\mathbf{3}\}$

$$(C \cup A)\backslash(B \cap D) = (N \cup \{-2, -1, 0\}) \backslash\emptyset = N \cup \{-2, -1, 0\}$$

$B - B =$

$C\backslash(A \cap D) =$

$(B \cup A)\backslash C =$

Theorem2.50: Let A, B and C any three sets. Then:

1. $A\backslash A = \emptyset, \quad A\backslash U = \emptyset$ (**H. W.**)
2. $A\backslash\emptyset = A, \quad \emptyset\backslash A = \emptyset$ (**H. W.**)
3. $A\backslash B \subseteq A, \quad B\backslash A \subseteq B$ (**H. W.**)
4. $A \subseteq B \Leftrightarrow A\backslash B = \emptyset$ (**H. W.**)
5. $A\backslash B \cap B = \emptyset$
6. $A \cap B = \emptyset \Leftrightarrow A\backslash B = A \wedge B\backslash A = B$
7. $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$ (**H. W.**)
8. $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$
9. $A\backslash A^c = A, \quad A^c\backslash A = A^c$ (**H. W.**)

Proof 5: Assume that $A\backslash B \cap B \neq \emptyset$ برهان غير مباشر

$\exists x, x \in A\backslash B \cap B \Longrightarrow x \in A\backslash B \wedge x \in B$

$\Longrightarrow (x \in A \wedge x \notin B) \wedge x \in B$ (def. of difference)

$\Longrightarrow x \in A \wedge (x \notin B \wedge x \in B)$ ($\wedge$ is asso.)

$\Longrightarrow x \in A \wedge F$

$\Longrightarrow x \in F$ تناقض $(p \wedge \mathrm{F} = F)$

$\therefore A\backslash B \cap B = \emptyset$

Proof 6: $(\Longrightarrow)$ Let $A \cap B = \emptyset$ T. P $A\backslash B = A \wedge B\backslash A = B$

Let $x \in A\backslash B \Leftrightarrow x \in A \wedge x \notin B$ (def. of \)

$\Leftrightarrow x \in A \wedge T$ (by hypo.)

$\Leftrightarrow x \in A$ $(\mathrm{p} \wedge T = p)$

$\therefore A\backslash B = A$

Similarly, one can prove that $B\backslash A = B$

$\Longleftarrow$) Let $A\backslash B = A \wedge B\backslash A = B$ T. P $A \cap B = \emptyset$

Assume that $A \cap B \neq \emptyset$ برهان بالتناقض

$\exists x, x \in A \cap B \Longrightarrow x \in A \wedge x \in B$ (def. of ∩)

$\Longrightarrow \exists x,\ x \in A\backslash B \wedge x \in B \quad (A\backslash B = A)$

$\Longrightarrow \exists x, x \in A\backslash B \wedge x \in B$

$\Longrightarrow \exists x, (x \in A \wedge x \notin B) \wedge x \in B$

$\Longrightarrow \exists x, x \in A \wedge (x \notin B \wedge x \in B)$ (∧ is asso.)

$\Longrightarrow \exists x, x \in A \wedge F = F$ تناقض $(p \wedge F = F)$

$\therefore A \cap B = \emptyset$

<u>Proof 8</u>: T. P $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$

let $x \in A\backslash(B \cap C) \Leftrightarrow x \in A \wedge x \notin B \cap C$ (def. of \)

$\Leftrightarrow x \in A \wedge (x \notin B \text{ v } x \notin C)$ (def. of ∩)

$\Leftrightarrow (x \in A \wedge x \notin B) \text{ v } (x \in A \wedge x \notin C)$ (dist. ∧ over v)

$\Leftrightarrow x \in A\backslash B \text{ v } x \in A\backslash C$ (def. of \)

$\Leftrightarrow x \in A\backslash B \cup A\backslash C$

$\therefore A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$

5. Symmetric Difference الفرق التناظري

The **symmetric difference** between two sets A and B is denoted by $A\Delta B$ and is defined as:

$$\boxed{\begin{aligned} A \,\Delta\, B &= (A\backslash B) \cup (B\backslash A) \\ &= (A \cup B) \backslash (A \cap B) \end{aligned}}$$

<u>Example2.51</u>: Let $A = \{x \in E: -8 \leq x < 9\} = \{-8, -6, \dots, 0, 2, \dots 8\}$

$B = \{1, 2, 4, 6\}$. Find $A\Delta B$

Solution: $A \, \Delta \, B = (A \backslash B) \cup (B \backslash A)$

$= \{-8, -6, \dots 0, 8\} \cup \{1\}$

Theorem2.52: Let $A, B \; and \; C$ any three sets. Then:

1. $A \Delta A = \emptyset$ $(\boldsymbol{H.W.})$, $A \, \Delta \, \emptyset = A$
2. $A \, \Delta \, B = B \, \Delta \, A$
3. $A \, \Delta \, B = \emptyset \iff A = B$
4. $A \, \Delta \, (B \, \Delta \, C) = (A \Delta B) \Delta \, C$ **(بدون برهان)**
5. $A \cap (B \, \Delta \, C) = (A \cap B) \Delta (A \cap C)$ **(بدون برهان)**

Proof 1: T. P $A \, \Delta \, \emptyset = A$

Let $x \in A \, \Delta \, \emptyset \iff x \in (A \backslash \emptyset \cup \emptyset \backslash A)$ (def. of Δ)

$\iff x \in A \backslash \emptyset \vee x \in \emptyset \backslash A$

$\iff x \in A \cup \emptyset$ $(A \backslash \emptyset = A \, , \emptyset \backslash A = \emptyset)$

$\iff x \in A$

$\therefore \; A \, \Delta \, \emptyset = A$

Proof 2: T. P $A \, \Delta \, B = B \, \Delta \, A$

Let $x \in A \, \Delta \, B \iff x \in (A \backslash B \cup B \backslash A)$ (def. of Δ)

$\iff x \in (B \backslash A \cup A \backslash B)$ ($\cup$ is comm.)

$\iff x \in B \, \Delta \, A$

$\therefore A \, \Delta \, B = B \, \Delta \, A$

Proof 3: $A \, \Delta \, B = \emptyset \iff A = B$

$(\Longrightarrow)$ let $A \, \Delta \, B = \emptyset$ T. P $A = B$

Suppose $A \neq B \Longrightarrow \exists$x$, x \in A \wedge x \notin B$

$\Longrightarrow \exists$x$, x \in A \backslash B$ (def. of \)

$\Longrightarrow \exists$x$, x \in A \backslash B$ v $x \in B \backslash A$ (T $\vee$ p=T)

$\Longrightarrow \exists$x$, x \in A \backslash B \cup B \backslash A$ (def. of $\cup$)

$\Longrightarrow \exists$x$, x \in A \, \Delta \, B = \emptyset$

$\Longrightarrow \exists$x$, x \in \emptyset$ contradiction

$\therefore \; A = B$

$\Longleftarrow$) suppose $A = B$ T. P $A \, \Delta \, B = \emptyset$

$$
\begin{aligned}
A \,\Delta\, B &= A \backslash B \;\cup\; B \backslash A \quad (\text{def. of } \Delta) \\
&= A \backslash A \;\cup\; A \backslash A \quad (A = B) \\
&= \emptyset \cup \emptyset = \emptyset
\end{aligned}
$$

Foundation Of Mathematics

أسس الرياضيات

الأستاذ: عبدالله محمود جاسم

CHAPTER THREE

Relations

قسم الرياضيات

المرحلة : الاول

السنة الدراسية:٢٠٢٥-٢٠٢٦

CHAPTER THREE

Relations

العلاقات

Chapter Three Contents:

Definition3.1: Ordered Pair الزوج المرتب

An **ordered pair** of elements *a* and *b is* denoted by *(a, b)* where *a* is called the first element and *b* is the second element.

ليكن a و b عنصراً . الزوج المرتب المتكون من a و b يرمز له بالرمز (a, b) حيث أن a يسمى مسقط أول للزوج المرتب والعنصر b يسمى المسقط الثاني.

Remark3.2: Let a, b, c *and* d be four elements. Then:

1. $(a, b) \neq (c, d)$ in general
2. $(a, b) = (c, d) \Leftrightarrow a = c \wedge \mathrm{b} = \mathrm{d}$
3. $(a, b) = (b, a) \Leftrightarrow a = b$

Cartesian Product الضرب الديكارتي

Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the **product**, or **Cartesian product**, of A and B and denoted by $\boldsymbol{A} \times \boldsymbol{B}$.

$$\boldsymbol{A} \times \boldsymbol{B} = \{(a, b): a \in A \ \wedge b \in B\}$$
$$(a, b) \in \ A \times B \Leftrightarrow \ a \in A \ \wedge b \in B$$
$$(a, b) \notin \ A \times B \Leftrightarrow \ a \notin A \ \vee \ b \notin B$$

Example3.3: Let $A = \{a, b, c\}$, $\ B = \{5,4\}$. Find

$A \times B$=

$B \times A$=

$A \times A$ =

$B \times B$=

Remark 3.4: If the number of a set A equals n and the number of a set B equals m. Then the number of the elements of $A \times B$ is nm.

Example3.5: Let $A = \{x \in N: x \leq 3\} = \{1,2,3\}$
$B = \{0,3\}$, $\ C = \{1\}$. Find

$A \times A$=

$B \times B =$

$C \times C$=

$B \times C$=

$(B \cap C) \times A$=

$(B \cup C) \times B$=

Is , $A \times B = B \times A$?

Theorem3.6: Let A, B, C and D be non empty sets. Then:

1. $A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$
2. $A \times B = B \times A \Leftrightarrow A = B$
3. $A \times (B \cap C) = (A \times B) \cap (A \times C)$ (**H. W.**)
4. $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (**H. W.**)
5. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
6. $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ (**H. W.**)

Proof 1: T. P. $\emptyset \times A = \emptyset$
Suppose that $\emptyset \times A \neq \emptyset$ نفرض العكس
$\exists\, (x, y) \in \emptyset \times A \Rightarrow x \in \emptyset \wedge y \in A$ (def. of $A \times B$)
$\Rightarrow$ F $\wedge$ $y \in A$
$\Rightarrow F \quad (F \wedge p = F)$ تناقض
$\therefore \emptyset \times A = \emptyset$
In the same way, prove that $A \times \emptyset = \emptyset$ (**H. W**)

Proof 2: ($\Rightarrow$) Suppose $A \times B = B \times A$ T. P. $A = B$
Let $(x, y) \in A \times B \Rightarrow x \in A \wedge y \in B \quad (def. of\ A \times B)$
$(x, y) \in B \times A \Rightarrow x \in B \wedge y \in A \quad (A \times B = B \times A)$
$\Rightarrow A \subseteq B \wedge B \subseteq A$
$\Rightarrow$ A = B (def. of equal sets)
$\Leftarrow$) Suppose $A = B$ T. P. $A \times B = B \times A$
Let $(x, y) \in A \times B \Leftrightarrow x \in A \wedge y \in B \quad (def. of\ A \times B)$
$\Leftrightarrow x \in B \wedge y \in A \quad (A = B)$
$\Leftrightarrow (x, y) \in B \times A$

$\therefore A \times B = B \times A$

Proof 5: T. P. $A \times (B \backslash C) = (A \times B) \backslash (A \times C)$
Let $(x, y) \in A \times (B \backslash C) \Leftrightarrow x \in A \wedge y \in (B \backslash C) \quad (def. of\ A \times B)$
$\Leftrightarrow x \in A \wedge (y \in B \wedge y \notin C) \quad (def. of\ \backslash)$
$\Leftrightarrow (x \in A \wedge y \in B) \wedge (x \in A \wedge y \notin C)$ (dist. $\wedge$ on $\wedge$)
$\Leftrightarrow (x, y) \in A \times B \wedge (x, y) \notin A \times C$
$\Leftrightarrow (x, y) \in (A \times B) \backslash (A \times C)$
$\therefore A \times (B \backslash C) = (A \times B) \backslash (A \times C)$

<u>Definition</u>3.7: Generalization of the Cartesian product تعميم الضرب الديكارتي

Let $A_1, A_2, \dots, A_n$ be any sets. Then

$$\prod_{i=1}^{n} A_i = A_1 \times \dots \times A_n = \{(x_1, \dots, x_n) : x_i \in A_i, i = 1, \dots, n\}$$

<u>Example</u>3.8: What is the Cartesian product $A \times B \times C$, where $A = \{0,1\}, B = \{1,2\}, C = \{2\}$?

<u>Solution</u>:

$A \times B \times C = \{(0,1,2), (0,2,2), (1,1,2), (1,2,2)\}$

<u>Remark</u>3.9: Let A be a set, then

$$A \times A = A^2$$
$$A \times A \times A = A^3$$

In general, $A \times \dots \times A = A^n$

<u>Example3.10</u>: Let $\mathbb{R}$ be the set of real numbers then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ and

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$

<u>Relation</u>: العلاقة

Let A and B are two sets. Any subset R of $A \times B$ is called a **relation** from A to B. In other words,

R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$.

$(x, y) \in R$ can be written as xRy or $x \sim y$
$(x, y) \notin R$ can be written as xRy or $x \nsim y$

Remark3.11: The relations are denoted by $R, S, T, W, \dots$

Definition3.12: If R is a relation from $\boldsymbol{A}$ ***to*** $\boldsymbol{A}$ ($R \subseteq A \times A$) then R is called a **relation on** $\boldsymbol{A}$

Example3.13 Let $A = \{1, 4, 5\}$, $B = \{1, a\}$

$A \times B = \{(1,1), (1, a), (4,1), (4, a), (5,1), (5, a)\}$. Write three relations from A to B.

Solution:

$R =$

$S =$

$T =$

Remark3.14: The empty set $\emptyset$ is the **smallest relation** from A to B

($\emptyset \subseteq A \times B$). And, $A \times B$ is the **largest relation** from A to B

($A \times B \subseteq A \times B$).

Example3.15: (H. W.) Let $A = \{1, 4, 5\}$, $B = \{1, a\}$. Find $B \times A$ and write three relations from B to A.

Example3.16: Let $A = \{x, y, -1\}$. Find a relation from A to A.
Solution: $A \times A = \{(a, , b): a, b \in A\}$

$A \times A =$

$\{(x, x), (x, y), (x, -1), (y, x), (y, y), (y, -1), -1, x), (-1, y), (-1, -1)\}$

<u>Specifying a relation</u>: طرق التعبير عن العلاقة

1. Listing members of a relation: الطريقة الجدولية

List the members of the relation separated by commas and contained in braces { }.

<u>Example</u>3.17: Let $A = \{1,2\}$, and $B = \{x, y\}$ are two sets and R is a relation from $A\ to\ B$.

R=$\{(1, x), (1, y), (2, x), (2, y)\}$

2. Listing a relation property: استخدام الصفة المميزة للعلاقة

State the property that characterizes the elements in a relation

<u>Example 3.18</u>: $A = \{-1,5,0\}$
$B = \{-3,0,-1\}$

Let

$R_1 = \{(a, b) \in A \times B: a \geq b\} =$
$\{(-1, -3), (-1, -1), (5, -3), (5,0), (5, -1),$ (0,-3),(0,0),(0,-1)}

$R_2 = \{(x, y) \in A \times B: x = y\} = \{(-1, -1), (0,0)\}$

<u>Definition 3.19</u>: Let $x\ and\ y$ are integers with $x \neq 0$ $(x, y \in Z)$. Then "x divides يقسم y" is denoted by $x|$y and is defined as:

$$\boxed{\begin{array}{l} x|y \iff \exists k \in Z \text{ such that } y = kx \\ x \nmid y \iff \forall k \in Z :\ y \neq kx \end{array}}$$

When "x divides يقسم y" we say that "x is a **factor** of y" or "y is a **multiple** of x"

<u>Example3.20</u>: Are $3|6, -4|8,\ -5|1,\ 9|30$?

<u>Solution</u>:

$\exists k = 2 \text{ such that } 6 = 3k = 3(2) \implies 3|6$

$\exists k = -2 \text{ such that } 8 = -4k = -4(-2) \implies -4|8$

$\forall k \in Z, 30 \neq 9k \implies 9|30$

$-5|1$ (**H. W.**)

Example3.21: $A = \{x \in Z: 0 \leq x \leq 4\}$. Write a relation R on A such that $R = \{(a,b) \in A \times A: a|b\}$

Solution: $A = \{0,1,2,3,4\}$

$R = \{(1,0),(1,1),(1,2),(1,3),(1,4),(2,0),(2,2),(2,4),(3,0),(4,0),(4,4)\}$

Definition3.22: Let R_1 and R_2 are two relations from A to B, then

$R_1 \cap R_2, R_1 \cup R_2, and\ R_1 \backslash R_2$ are also relations from A to B.

Example3.23: $A = \{x, y, z\}$
$B = \{x \in Z: -2 \leq x \leq 3\}$
Write two relations R_1 and R_2 from A to B.

R_1=

R_2=

Then find

$R_1 \cap R_2$=

$R_1 \cup R_2$=

$R_1 \backslash R_2$=

$R_2 \backslash R_1$=

Definition3.24: Let R a relation from A to B then R^{-1} is a relation from $\boldsymbol{B}$ **to** $\boldsymbol{A}$. R^{-1} is called **the inverse** of R.

$$R^{-1} = \{(b,a):(a,b) \in R\}$$

Remark 3.25: $(R^{-1})^{-1} = R$

Example 3.26: Let $A = \{0,3,8,-10\} and\ B = \{0,1,2\}$. A relation R from A to B is defined as:

$$R = \{(a,b) \in A \times B : a+b = 2k, k \in Z\}$$

Write the elements of the relation R then find R^{-1}?

Definition3.27: Domain of a relation مجال العلاقة

The domain of a relation $R \subseteq A \times B$ is the set of the first coordinates of each pair**.** In other words:

dom $R = \{x \in A;\ \exists y \in B : (x,y) \in R\}$

It is clear that $domR \subseteq A$

منطلق العلاقة هو مجموعة المساقط الأولى للعلاقة

Definition 3.28: Range of a relation مدى العلاقة

The range of a relation $R \subseteq A \times B$ is the set of the second coordinates of each pair**.** In other words:

range $R = \{y \in B;\ \exists x \in A : (x,y) \in R\}$

It is clear that $rangeR \subseteq B$

مدى العلاقة هو مجموعة المساقط الثانية

Example 3.29: Let $A = N\ and\ R = \{(a,b) \in N \times N : b = 2a\}$

Find $dom\ R\ and\ range\ R$.

Solution:

The relation R can be written as follows

$R = \{(1,2),(2,4),(3,6),(4,8),\ldots.\}$

$domR = \{1,2,3,4,5, \ldots .\} = N$

$range\ R = \{2,4,6,8, \ldots .\} =$ even positive numbers

<u>Example 3.30</u>: (H. W.) Let $A = Z\ and\ R = \{(a,b) \in Z \times Z : b = 1 - a\}$

Find $dom\ R\ and\ range\ R$.

<u>Lemma 3.31</u>: Let R be a relation on $A \times B$ then:

1. $dom\ R = range\ R^{-1}$ **(H. W.)**

2. $range\ R = dom\ R^{-1}$

Proof 2: $A \times B = \{(a,b) : a \in A, b \in B\}$

$B \times A = \{(b,a) : (a,b) \in A \times B\}$

$rangeR = \{b \in B; \exists a \in A\ such\ that\ (a,b) \in R\}$

$domR^{-1} = \{b \in B; \exists a \in A\ such\ that\ (b,a) \in R^{-1}\}$

T. P $range\ R = dom\ R^{-1}$

Let $b \in rangeR \quad \Leftrightarrow \exists a \in A\ such\ that\ (a,b) \in R$

$\Leftrightarrow \exists a \in A\ such\ that\ (b,a) \in R^{-1}$

$\Leftrightarrow b \in dom\ R^{-1}$

$\therefore$ $range$ R= dom R^{-1}

<u>Properties of Relations</u>: خواص العلاقات

1. Reflexive relation: العلاقة الانعكاسية

A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every $a \in A$

العلاقة Rعلى المجموعة A تسمى انعكاسية اذا كان كل عنصر في المجموعة A مرتبط بنفسه وفق العلاقة R.

R **reflexive** on $A \Leftrightarrow a\,R\,a \quad \forall a \in A$

$\Leftrightarrow (a,a) \in R \quad \forall a \in A$

R **not reflexive** on $A \Leftrightarrow \exists a \in A,\ a \nsim a$

$\Leftrightarrow \exists a \in A,\ (a,a) \notin R$

<u>Example 3.32</u>: Let $A = \{1,2,3,4\}$. Which of these relations are reflexive?

<u>Solution</u>:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,1),(4,4)\}$

R_1 is reflexive on A because

$(1,1) \in R_1, (2,2) \in R_1, (3,3) \in R_1, (4,4) \in R_1$

$\Longrightarrow\ (a,a) \in R_1 \quad \forall a \in A$

$R_2 = \{(1,1),(1,2),(2,1)\}$ is not reflexive because $\exists 2 \in A\ such\ that$

$(2,2) \notin R_2$

$\exists 3 \in A\ such\ that\ (3,3) \notin R_2$

$\exists 4 \in A\ such\ that\ (4,4) \notin R_2$

$R_3 = \{(1,1),(2,1),(3,4),(2,2),(2,3),(3,3),(3,1),(3,1),(4,3)\}$ is not reflexive because $(4,4) \notin R_3$

$R_4 = \{(3,4)\}$ is not reflexive

<u>Example 3.33</u>: Let $A = \{-2, \frac{1}{2}, 0, 3\}$. Let R_1 and R_2 be two relations on A such that

$R_1 = \{(a,b) : a \leq b\}$

$R_2 = \{(a,b): a = b\}$

Are R_1, R_2 reflexive? Is $A \times A$ reflexive on A?

Solution: $R_1 = \{(a,b): a \leq b\} = \{(-2,-2), \left(-2, \frac{1}{2}\right), (-2,0), (-2,3),$

$\left(0, \frac{1}{2}\right), (0,3), \left(\frac{1}{2}, 3\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0,0), (3,3)\}$

$(a,a) \in R_1 \quad \forall a \in A \Longrightarrow R_1$ is reflexive

Similarly, $(a,a) \in R_2 \quad \forall a \in A$ and $(a,a) \in A \times A \; \forall a \in A$

$\Longrightarrow R_2$ and $A \times A$ are reflexive relations

Example 3.34: Let $A = Z$. Let R be relations on A such that

$R = \{(a,b): a = b \; or \; a = -b\}$.

Is R reflexive? Is $Z \times Z$ reflexive on Z?

Solution:

$R = \{(-1,-1), (-1,1), (1,-1), (0,0), (2,-2), \ldots.\}$

Since $(a,a) \in R \quad \forall a \in Z \Longrightarrow R$ is reflexive

Similarly, $(a,a) \in Z \times Z \quad \forall a \in Z \Longrightarrow Z \times Z$ is reflexive

Remark 3.35:

1. $A \times A$ is reflexive on A

2. $\emptyset$ is not reflexive on A

Example3.36: Let $A = N$. Let R be relations on A such that

$R = \{(a,b) \in N \times N: a|b\}$. Is R reflexive?

Solution:

$a\ R\ a \quad \forall a \in N?\ Is\ a|a\ \forall a \in N?$

$a|a \Longrightarrow\ \exists k = 1\ s.t.\ a = 1(a)$

$\Longrightarrow\ a\ R\ a \quad \forall a \in N$

R is reflexive

Example 3.37: Let $A = \{-2, -3, 2, 4\}$. Let R be relations on A such that

$R = \{(a, b): a + b \leq 3\}$. Is R reflexive?

Solution: Let $a = 2$, $b = 2$

$$a + b = 4 > 3$$

$\Longrightarrow R$ is not reflexive

Identity Relation: العلاقة الذاتية او علاقة الوحدة

Let A be a set. The **identity** relation on A is denoted by I_A and is defined as:

$$I_A = \{(a, b) \in A \times A: a = b\}$$

Remark 3.38: Let A be a set. The identity relation I_A is a subset of the reflexive relation on A.

Example 3.39: Let $A = \{-2, -3, 2, 4\}$. The following relation is Identity relation

$R = \{(-2, -2), (-3, -3), (2,2), (4,4)\} = I_A$

Example 3.40: (**H. W.**) Which relations from **examples 3.32-3.34, 3.36, 3.37** are identity relations?

Symmetric Relation: العلاقة التناظرية

A relation R on a set A is called **symmetric** if the following condition satisfied:

$$\boxed{\text{If } (a,b) \in R \ then \ (b,a) \in R \quad \forall a,b \in A}$$

The relation R is **not symmetric** غير تناظرية if

$$\boxed{\exists (a,b) \in A \times A \text{ such that } (a,b) \in R \ but \ (b,a) \notin \mathrm{R}}$$

Example 3.41: Let $A = \{1,2,3,4\}$. Which of these relations are symmetric?

$R_1 = \{(1,2),(2,1)\}$ is symmetric because $(1,2) \in R_1 \wedge (2,1) \in R_1$

$R_2 = \{(1,1),(1,2),(2,1)\}$ is symmetric

$R_3 = \{(1,1),(3,4),(2,2),(2,3),(3,3),(3,1),(1,3),(4,3)\}$ is not symmetric because $(2,3) \in R_3$ but $(3,2) \notin \mathrm{R}_3$

$R_4 = \{(3,4)\}$ is not symmetric

Example 3.42: Let $A = \{-2,-3,2,4\}$. Let R be relations on A such that

$R = \{(a,b): a + b \leq 3\}$. Is R symmetric?

Solution:

$$R = \{(-2,-2),(-2,-3),(-2,2),(-2,4),(-3,-2),(-3,2),(-3,4),$$
$$(2,-2),\ (2,-3),(4,-2),(4,-3)\}$$

R is symmetric, $(a,b) \in R \Leftrightarrow (b,a) \in R$

Example 3.43: Let $A = Z$. Let R be relations on A such that

$R = \{(a, b) \in Z \times Z: a + b \leq 3\}$. Is R symmetric?

Solution: $R = \{(-1,-1), (-2,-2), (2,1), (1,2), (1,1), \dots\}$

نلاحظ في هذا المثال ان العلاقة Rمعرفة على مجموعة الاعداد الصحيحة الغير منتهية وانه من غير الممكن كتابة جميع عناصر العلاقة R. لذلك فانه من الصعب اختبار خاصية التناظر من خلال كتابة كل عناصر العلاقة. لذلك سنلجأ الى طريقة الاختبار عن طريق البرهان العام كالتالي:

$(a, b) \in R \Leftrightarrow a + b \leq 3$ **or** $a\, R\, b \Leftrightarrow a + b \leq 3$

Let $(a, b) \in R \Longrightarrow (b, a) \in R$?

$$(a, b) \in R \Longrightarrow a + b \leq 3$$

$$\Longrightarrow b + a \leq 3$$

$$\Longrightarrow (b, a) \in R$$

$\therefore$ R is symmetric relation

Example 3.44: Let $A = Z$ and define $a\, R\, b \Leftrightarrow ab \geq 0 \quad \forall a, b \in Z$

Is R reflexive? Symmetric?

Solution:

reflexive: Is $aRa \quad \forall a \in Z$?

Let $a \in Z \Longrightarrow \mathrm{a.a} = \mathrm{a}^2 \geq 0$

$\therefore$ R is reflexive

symmetric? Let $(a, b) \in R, Is\ (b, a) \in R$?

$$(a, b) \in R \Longrightarrow ab \geq 0$$

$$\Longrightarrow ba \geq 0$$

$$\Longrightarrow (b, a) \in R$$

$\therefore$ R is symmetric

Example 3.45: Let $A = \mathbb{R}$ and define $a\ S\ b \Leftrightarrow a - b > 0 \quad \forall a, b \in S$

Is S reflexive? Symmetric?

Solution:

1. reflexive: Is $aSa \quad \forall a \in \mathbb{R}$?

 Let $a \in R \Longrightarrow \mathrm{a} - \mathrm{a} = 0$

$\therefore S$ is not reflexive

2. symmetric? Let $(a, b) \in S, Is\ (b, a) \in S$?

$$(a, b) \in S \Longrightarrow a - b > 0$$

$$\Longrightarrow b - a < 0$$

$$\Longrightarrow (b, a) \notin \mathrm{S}$$

$\therefore S$ is not symmetric

Example 3.46: (**H. W.**) Let $A = Z$ and define $a\ R\ b \Leftrightarrow |a| = |\mathrm{b}| \quad \forall a, b \in Z$

Is R reflexive? Symmetric?

Example 3.47: (**H. W.**) Let $A = Z$ and define $a\ R\ b \Leftrightarrow \mathrm{a} = 1 \ \ \forall a \in Z$

$R = \{(1, b) : b \in Z\}$

Is R reflexive? Symmetric?

Theorem 3.48: A relation R on a set A is symmetric iff $R = R^{-1}$

Proof: $\Longrightarrow$) Suppose R is symmetric T. P. $R = R^{-1}$

Let $(a, b) \in R \Leftrightarrow (b, a) \in R$ (R is symmetric)

$\Leftrightarrow (a, b) \in R^{-1}$ (def. of R^{-1})

$\therefore R = R^{-1}$

$\Longleftarrow$) Suppose $R = R^{-1}$ T. P. R is symmetric

Let $(a, b) \in R \Longrightarrow (a, b) \in R^{-1}$ $(R = R^{-1})$

$\Longrightarrow (b, a) \in (R^{-1})^{-1} = R$

$\therefore R$ is symmetric.

<u>Anti-Symmetric Relation</u>: علاقة ضد التناظرية

A relation R on a set A is called **anti symmetric** if

$$(a\,R\,b\ \wedge \mathrm{b}\ Ra) \Longrightarrow a = b\ \ \forall a, b \in A$$

R is not **anti-symmetric** if $\exists a, b \in A\ such\ that\ (a\ Rb\ \wedge \mathrm{b}\ Ra) \wedge \mathrm{a} \neq b$

<u>Example 3.49</u>: Let $A = \{1,2,3,4\}$ and R_1, R_2 are two relations on A such that

$R_1 = \{(2,1), (3,1), (3,2), (1,1)\}$ and $R_2 = \{(2,1), (3,1), (1,2), (1,1)\}$

Are R_1, R_2 anti symmetric?

<u>Solution</u>: $R_1$is anti symmetric because $(1\ R\ 1\ \wedge 1\ R1) \Longrightarrow 1 = 1$

R_2 is not anti symmetric because $\exists(2,1) \in R \wedge (1,2) \in R$ but $1 \neq 2$

<u>Example 3.50</u>: (H. W.) Let $A = \{1,2,3,4\}$ and $R$is a relation on A such that

$R = \{(4,2)\}$. Is R anti symmetric?

<u>Example3.51</u>: Let $A = Z$ and R is a relation on Z such that $a\ R\ b \iff a = b + 1$

Is R reflexive? symmetric? Anti-symmetric?

<u>Solution</u>:

$R = \{(2,1), (1,0), (0,-1), \dots.\}$

Reflexive? Let $(a, a) \in R \Longrightarrow \mathrm{a} \neq a + 1$

$\therefore R$ is not reflexive

symmetric? Let $(a, b) \in R \Longrightarrow a = b + 1$

but $b \neq a + 1$

Take $(3,2) \in R$ $(3 = 2 + 1)$

but $(2,3) \notin R$ $(2 \neq 3 + 1)$

$\therefore$ R is not symmetric

anti-symmetric? Suppose $a\ R\ b\ \wedge b\ R\ a \ \Longrightarrow a = b$?

if $a\ R\ b$ then $a = b + 1$

if $b\ R\ a$ then $b = a + 1$

$a = b + 1$ and $b = a + 1$ iff $a = b$

$\therefore R$ is anti symetric

<u>Example 3.52</u>: Let $A = Z$ and R is a relation on Z such that $a\ R\ b \iff a|b$

Is R anti-symmetric?

<u>Solution</u>:

Let $a\ R\ b\ \wedge b\ R\ a \ \Longrightarrow a = b$?

$$a\ R\ b\ \wedge b\ R\ a \ \Longrightarrow\ a|b \wedge\ b|a$$

$$\Longrightarrow b = k_1 a\ \wedge a = k_2 b\ ,\ \ k_1, k_2 \in Z \ldots.(*)$$

$$\Longrightarrow b = k_1(k_2 b)$$

$$\Longrightarrow b = (k_1 k_2) b$$

$$\Longrightarrow k_1 k_2 = 1$$

$\Longrightarrow \mathrm{k}_1 = k_2 = 1\ or\ k_1 = k_2 = -1$

If $\mathrm{k}_1 = k_2 = 1\ then\ b = 1.a$ (from *)

If $k_1 = k_2 = -1$ $then\ b = -1.a$ (from *)

$\Longrightarrow b = a\ or\ b = -a$

$\therefore$ R is not anti symmetric.

Example 3.53: (H. W.) Let $A = Z$ and R is a relation on Z such that

$$a\ R\ b \iff a + b = 2k, \qquad k \in Z$$

Is R anti-symmetric?

Theorem 3.54: Let R be a relation on A, then R is anti symmetric iff $R \cap R^{-1} \subseteq I_A$

Proof: $\Rightarrow$) Let R is anti symmetric **T. P.** $R \cap R^{-1} \subseteq I_A$

Let $(a, b) \in R \cap R^{-1} \Longrightarrow (a, b) \in R \wedge (a, b) \in R^{-1}$ (def. of $\cap$)

$\Longrightarrow (a, b) \in R \wedge (b, a) \in R$

$\Longrightarrow a = b$ (R is anti symmetric)

$\Longrightarrow (a, b) \in I_A$

$\therefore R \cap R^{-1} \subseteq I_A$

$\Leftarrow$) Let $R \cap R^{-1} \subseteq I_A$ **T. P.** R is anti symmetric

Let $aRb \wedge b\ Ra \Longrightarrow (a, b) \in R \wedge (b, a) \in R$

$\Longrightarrow (a, b) \in R \wedge (a, b) \in R^{-1}$

$\Longrightarrow (a, b) \in R \cap R^{-1}$ (def. of $\cap$)

$\Longrightarrow (a, b) \in I_A$

$\Longrightarrow a = b$

Transitive Relation: العلاقة المتعدية

A relation R on a set A is **transitive**

If $(a,b) \in \mathrm{R} \wedge (b,c) \in \mathrm{R}$ then $(a,c) \in \mathrm{R}\ \forall a,b,c \in A$

or

If $aR\ b\ \wedge\ b\ R\ c$ then $a\ R\ c\ \ \forall a,b,c \in A$

A relation R on a set A is **not transitive** if

$\exists a,b,c \in A\ such\ that\ (a,b) \in R \wedge (b,c) \in R\ \wedge (a,c) \notin R$

<u>Example 3.55</u>: Let $A = \{1,2,3,4\}$. Which of these relations are transitive?

$R_1 = \{(1,1),(1,2),(2,3),(1,3)\}$

R_1 is transitive on A because

$(1,2) \in R_1 \wedge (2,3) \in R_1 \Longrightarrow\ (1,3) \in R_1$

$(1,1) \in R_1 \wedge (1,2) \in R_1 \Longrightarrow\ (1,2) \in R_1$

$R_2 = \{(1,2),(2,3)\}$

R_2 is not transitive on A because

$(1,2) \in R_2 \wedge (2,3) \in R_2$ but $(1,3) \notin R_2$

<u>Example 3.56</u>: Let $A = N$. Define a relation R on A such that

$R = \{(a,b): a \leq b\}$. Is R transitive on A? symmetric on A?

<u>Solution</u>:

Transitive? Let $(a,b) \in R \wedge (b,c) \in R$, Is $(a,c) \in R$?

$(a, b) \in R \wedge (b, c) \in R \Longrightarrow a \leq b \wedge b \leq c$

$$\Longrightarrow a \leq c \Longrightarrow (a, c) \in R$$

$\therefore R$ is transitive on A

symmetric? Let $(a, b) \in R$, Is $(b, a) \in R$?

Let $(a, b) \in R \Longrightarrow a \leq b$

but this does not mean that $b \leq a$

for example, let a=2, b=5

$2 \leq 5$ but $5 > 2$

$\therefore R$ is not symmetric on A

<u>Example 3.57</u>: Let $A = N$. Define $a\ R\ b \Leftrightarrow a|\text{b}$

Prove that R **transitive** and **not symmetric** on A

<u>Solution</u>:

transitive: let $a, b, c \in N$ such that $a\ R\ b \wedge b\ R\ c$ To Prove $a\ R\ c$

$a\ R\ b \Longrightarrow a|\text{b}$

$\Longrightarrow \exists k_1 \in Z\ s.t.\ b = k_1 a$ **.....(1)**

$b\ R\ c \Longrightarrow b|\text{c}$

$\Longrightarrow \exists k_2 \in Z\ s.t.\ c = k_2 b$ **.....(2)**

substitute (1) into (2), $c = k_2 k_1 a$

$$\Longrightarrow c = k_3 a, \quad k_3 = k_1 k_2 \in Z$$

$$\Longrightarrow a|\text{c}$$

$$\Longrightarrow a\ R\ c$$

$\therefore R$ is transitive

R is not symmetric. Take a=2 and b=4

It is clear that 2|4 (4=2(2))

but "4 does not divides 2" $\Longrightarrow \forall k \in Z, 2 \neq 4k$

<u>Example 3.58</u>: (**H. W.**) Let $A = Z$ and define $a\ R\ b \Leftrightarrow ab \geq 0 \quad \forall a, b \in Z$

Show that R is transitive.

<u>Example 3.59</u>: (**H. W.**) Let $A = R$ and define

$a\ S\ b \Leftrightarrow a - b > 0\ \forall a, b \in R$. Is S transitive?

<u>Equivalence Relation</u>: علاقة التكافؤ

A relation R on a set A is called equivalence relation if and only if R is reflexive, symmetric and transitive

العلاقة التي تكون انعكاسية وتناظرية ومتعدية تسمى علاقة تكافؤ

<u>Example 3.60</u>: Let $A = Z$ and R is a relation on Z such that

$$a\ R\ b \Leftrightarrow a + b = 2k, \qquad k \in Z$$

Show that R is equivalence relation?

<u>Solution</u>:

reflexive: T. P. $a\ R\ a \quad \forall a \in Z$

let $a \in Z \Longrightarrow a + a = 2a \Longrightarrow (\mathrm{a}, \mathrm{a}) \in \mathrm{R}$

Symmetric? Let $(a, b) \in R$ T. P. $(b, a) \in R$

$(a, b) \in R \Longrightarrow a + b = 2k$

$\Longrightarrow \mathrm{b} + a = 2k$

$\Longrightarrow (b, a) \in R$

$\therefore R$ is Symmetric

Transitive: Let $(a,b) \in R \wedge (b,c) \in R$ To Prove $(a,c) \in R$

$(a,b) \in R \Longrightarrow a+b=2k_1,\ \ k_1 \in Z$**……(1)**

$(b,c) \in R \Longrightarrow b+c=2k_2,\ \ k_2 \in Z$**……(2)**

By summing up equations (1) and (2)

$$a+b+b+c=2k_1+2k_2$$

$$a+c=2(k_1+k_2-b)$$

$$a+c=2s \quad s=k_1+k_2-b \in Z$$

$$(a,c) \in R$$

$\therefore R$ is transitive

<u>Example 3.61</u>:(H.W.) Which relations from previous examples are equivalence relations?

<u>Equivalence Classes</u>: صفوف التكافؤ

Let R be an equivalence relation on A. The set of all elements that are related to an element $a \in A$ is called an **equivalence class** of a. The equivalence class of a is denoted by $[a]$.

لتكن R علاقة تكافؤ معرفة على A وليكن aعنصرا في A. فان مجموعة كل العناصر التي ترتبط وفق العلاقة R مع a تسمى صف تكافؤ ال a ويرمز لها ب $[a]$.

$[a]=\{x \in A: x\sim a\}$

$x \in [a] \Leftrightarrow x\sim a$

$x \notin [a] \Leftrightarrow x \nsim a$

<u>Example</u>3.62 Let $A=\{1,2,3,4\}$. $R=\{(1,1),(2,2),(1,2),(2,1),(3,3),(4,4)\}$ be an equivalence relation on A. Find all equivalence classes on A.

Solution:

$[1] = \{x \in A: x \sim 1\} = \{1,2\}$

$[2] = \{x \in A: x \sim 2\} = \{1,2\}$

$[3] = \{x \in A: x \sim 3\} = \{3\}$

$[4] = \{x \in A: x \sim 4\} = \{4\}$

Example3.63: Let $A = \{-1,1,0\}$ and $R = \left\{(a,b) \in A \times A: \sqrt[3]{a} = \sqrt[3]{b}\right\}$. Show that R is an equivalence relation. Find all equivalence classes on A.

Solution: $R = \{(-1,-1),(1,1),(0,0)\}$

$[-1] = \{x \in A: x \sim -1\} = \{-1\}$

$[1] = \{x \in A: x \sim 1\} = \{1\}$

$[0] = \{x \in A: x \sim 0\} = \{0\}$

Example3.64: Let $A = Z$ and $R = \{(a,b) \in Z \times Z: a - b = 3k,\ k \in Z\}$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

Solution:

reflex.: Let $a \in Z$ T.P. $(a,a) \in R$

$$a \in Z \Longrightarrow a - a = 3(0), \quad k = 0 \in Z$$

Symm.: Let $(a,b) \in R \Longrightarrow a - b = 3k, \quad k \in Z$

$$\Longrightarrow b - a = -3k = 3(-k), \quad -k \in Z$$

$$\Longrightarrow (b,a) \in R$$

Trans.: Let $(a,b) \in R \wedge (b,c) \in R$ To Prove $(a,c) \in R$

$(a,b) \in R \Longrightarrow a - b = 3k_1,\ \ k_1 \in Z$**……(1)**

$(b,c) \in R \Longrightarrow b - c = 3k_2,\ \ k_2 \in Z$**……(2)**

By summing up equations (1) and (2)

$$a - b + b - c = 3k_1 + 3k_2$$

$$a - c = 3(k_1 + k_2)$$

$$a - c = 3s \quad s = k_1 + k_2 \in Z$$

$$(a, c) \in R$$

$\therefore R$ is reflexive, symm. and trans.

$\therefore R$ is equivalence relation on Z

To find all equivalence classes, we start with

$[0] = \{x \in Z : x \sim 0\} = \{x \in Z : x - 0 = 3k,\ k \in Z\}$

$= \{x \in Z : x = 3k,\ k \in Z\}$

$=\{0,3,-3,6,-6,\ldots..\}$

$[1] = \{x \in Z : x \sim 1\} = \{x \in Z : x - 1 = 3k,\ k \in Z\}$

$= \{x \in Z : x = 3k + 1,\ k \in Z\}$

$= \{1,4,-2,7,-5,\ldots..\}$

$[2] = \{x \in Z : x \sim 2\} = \{x \in Z : x - 2 = 3k,\ k \in Z\}$

$= \{x \in Z : x = 3k + 2,\ k \in Z\}$

$= \{2,5,-1,8,-4,\ldots..\}$

$[3] = \{x \in Z : x \sim 1\} = \{x \in Z : x - 3 = 3k,\ k \in Z\}$

$= \{x \in Z : x = 3k + 3,\ k \in Z\}$

$= \{3,6,0,9,-3,\ldots..\}=[0]$

$[4] = [1]$

$[5] = [2]$

<u>Example</u>3.65: (H. W.) Let $A = Z$ and $R = \{(a, b) \in Z \times Z: a - b = 5k,$ $k \in Z$. Show that R is an equivalence relation. Find all different equivalence classes on Z.

<u>Example</u>3.66: Let $A = N$. Define a relation R on A such that

$R = \{(a, b): a = b\}$. Show that R is an equivalence relation. Find all equivalence classes on N.

<u>Solution</u>:

R is an equivalence relation (**H. W.**)

$[1] = \{x \in N: x \sim 1\} = \{x \in N: x = 1\} = \{1\}$

$[2] = \{x \in N: x \sim 2\} = \{x \in N: x = 2\} = \{2\}$

$[3] = \{x \in N: x \sim 3\} = \{x \in N: x = 3\} = \{3\}$

……. etc

<u>Theorem3.67</u>: Let R be an equivalence relation on A, then:

1. $[a] \neq \emptyset \quad \forall a \in A$

2. $a \sim b$ if and only if $[a] = [b]$

3. $a \nsim b \iff [a] \cap [b] = \emptyset$

4. $[a] \neq [b] \iff [a] \cap [b] = \emptyset$

5. $a \in [b] \iff [a] = [b]$

<u>Proof1</u>: $[a] = \{x \in A: x \sim a\}$

Since R is an equivalence relation $\Longrightarrow R$ is reflexive

$$\Longrightarrow a \sim a \quad \forall a \in A$$

$\Longrightarrow a \in [a]$ (def. of equi. classes)

$\Longrightarrow [a] \neq \emptyset$

<u>Proof2</u>: $\Longrightarrow$) Suppose $a \sim b$ T. P $[a] \subseteq [b] \wedge [b] \subseteq [a]$

let $x \in [a] \Longrightarrow x \sim a \wedge a \sim b$

$\Longrightarrow x \sim b$ (R is trans.)

$\Longrightarrow x \in [b]$

$\therefore [a] \subseteq [b]$

Similarly, prove that $[b] \subseteq [a]$ (H. W.)

$\Longleftarrow$) Let $[a] = [b]$ T. P. $a \sim b$

since R is reflexive $\Longrightarrow a \sim a \Longrightarrow a \in [a] = [b]$ (from hypo.)

$\Longrightarrow a \in [b] \Longrightarrow a \sim b$

<u>Proof3</u>: $\Longrightarrow$) Suppose $a \nsim b$ T. P $[a] \cap [b] = \emptyset$

suppose $[a] \cap [b] \neq \emptyset$

$\exists x \in [a] \cap [b] \Longrightarrow x \in [a] \wedge x \in [b]$

$\Longrightarrow x \sim a \wedge x \sim b$ (def. of equi. classes)

$\Longrightarrow a \sim x \wedge x \sim b$ (R is symm.)

$\Longrightarrow a \sim b$ (R is trans.) contradiction

$\therefore [a] \cap [b] = \emptyset$

($\Longleftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $a \nsim b$

suppose $a \sim b \Longrightarrow [a] = [b]$ (from 2)

$\Longrightarrow [a] \cap [b] \neq \emptyset$ contradiction

$\therefore a \nsim b$

<u>Proof4</u>: $\Longrightarrow$) suppose $[a] \neq [b]$ and $[a] \cap [b] \neq \emptyset$

$\Longrightarrow a \sim b$ (from 3)

$\Longrightarrow [a] = [b]$ تناقض مع الفرض

$\Longleftarrow$) suppose $[a] \cap [b] = \emptyset$ T. P. $[a] \neq [b]$

suppose $[a] = [b] \Longrightarrow [a] \cap [b] \neq \emptyset$ تناقض

$\therefore [a] \neq [b]$

<u>Definition</u>: Partition of a Set تجزئة المجموعة

A collection of subsets $\{A_i : i \in I \subseteq N\}$ of A is called **partition** of A if

تجمع من المجموعات الجزئية الغير خالية$\{A_i : i \in N\}$ من A تسمى **تجزئة** ل A اذا حققت الشروط التالية

1. $A_i \neq \emptyset \quad \forall i \in I$

2. $A_i \cap A_j = \emptyset \quad \forall\, i \neq j$

3. $\bigcup_{i \in I} A_i = A$

<u>Example</u>3.68: Let $A = \{0,1,2,3,5,-2\}$. Find two partitions of A.

<u>Solution</u>: First partition

The collection $A_1 = \{0,1\}, A_2 = \{2,3,5\}, A_3 = \{-2\}$ form a partition of A

because

1. $A_1 \neq \emptyset, A_2 \neq \emptyset$ and $A_3 \neq \emptyset$

2. $A_1 \cap A_2 = \emptyset,\ A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$

3. $A_1 \cup A_2 \cup A_3 = A$

Second Partition(H.W.)

Example3.69: Let $X = [-2,5)$. Find two partitions of X.

Solution: First partition

The collection $A_1 = [-2,3], A_2 = (3,4), A_3 = [4,5)$ form a partition of X

Second Partition (H.W.)

Theorem3.70: Let R be an equivalence relation on a nonempty set A. Then the set of all different equivalence classes forms a partition for A.

Proof: Let $P = \{[a]: a \in A\}$ the set of all different equivalent classes of A

T. P. P is a partition of A

(1) $[a] \neq \emptyset \quad \forall a \in A$ (from Theorem 3.67(1))

(2) Let $[a], [b]$ are two different equivalence classes **T. P.** $[\boldsymbol{a}] \cap [\boldsymbol{b}] = \emptyset$

Since $[a] \neq [b] \Longrightarrow [\boldsymbol{a}] \cap [\boldsymbol{b}] = \emptyset$ (from Theorem 3.67(3))

(3) T.P. $\bigcup_{a \in A}[a] = A$

T. P. $\bigcup_{a \in A}[a] \subseteq A \wedge A \subseteq \bigcup_{a \in A}[a]$

let $x \in \bigcup_{a \in A}[a] \Longrightarrow x \in [a]$ for some $a \in A$

$\Longrightarrow x \in A \quad ([a] \subseteq A)$

$\therefore \bigcup_{a \in A}[a] \subseteq A$ **………(1)**

let $x \in A \Longrightarrow x \in [a]$ for some $a \in A$

$\Longrightarrow x \in \bigcup_{a \in A}[a] \quad ([a] \subseteq \bigcup_{a \in A}[a])$

$\therefore A \subseteq \bigcup_{a \in A}[a]$ **…….(2)**

From (1) and (2), $\bigcup_{a \in A}[a] = A$

Example3.71: Let $A = Z$ and define $a\, R\, b \Leftrightarrow |a| = |\mathrm{b}| \quad \forall a, b \in Z$

Prove that R is an equivalence relation. Then find all different equivalence classes (i.e., find a partition set of Z).

<u>Solution</u>: R is an equivalence relation **(H. W.)**

$[0] = \{x \in Z : x \sim 0\} = \{0\}$

$[1] = \{1, -1\}$

$[2] = \{2, -2\}$

.

.

etc

Partition set=P=$\{[a]: a \in Z^+\}$

<u>Example</u>3.72: Let $A = Z$ and $R = \{(a, b) \in Z \times Z : a - b = 3k,\ k \in Z\}$**.** Find the partition set on Z.

<u>Solution</u>: From Example 3.64, the relation R is an equivalence relation

Partition set=The set of all different equivalence classes

P= $\{[0], [1], [2]\}$

<u>Example</u>3.73: Let $A = \{1,2,3,4,5,6\}$ and
$A_1 = \{1,2,3\}, A_2 = \{4,5\}, and\ A_3 = \{6\}$ are partition of A. Write the equivalence relation R produced from A_1, A_2 and A_3.

<u>Solution</u>: The subsets A_1, A_2 and A_3 are the equivalence classes of A

1. $A_1 = \{1,2,3\}$ is an equivalence class of A

$\Longrightarrow (1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3) \in R$

2. $A_2 = \{4,5\}$ is an equivalence class of A

$\Longrightarrow (4,5), (4,4), (5,4), (5,5) \in R$

3. $A_3 = \{6\}$ is an equivalence class of A

$\Longrightarrow (6,6) \in R$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,5), (4,4), (5,4), (5,5), (6,6)\}$

Order Relations علاقات الترتيب

1. Partially Ordered Relation: العلاقة المرتبة جزئيا

A relation R on a set A is called **Partially Ordered Relation (P. O. R)** or **partially ordering if** it is reflexive, anti-symmetric and transitive. The pair (A, R) is called **partially ordered set**.

العلاقة على المجموعة A تسمى مرتبة جزئيا اذا كانت العلاقة انعكاسية وضد تناظرية ومتعدية كما يسمى الزوج (A, R) بالمجموعة المرتبة جزئيا.

Mathematically,

R is P.O.R $\Longleftrightarrow$ R reflexive $\wedge$ anti symmetric $\wedge$ transitive
R is not P.O.R $\Longleftrightarrow$ R not reflexive $\vee$ not anti symmetric $\vee$ not transitive

Example3.74: (**H.W.**) Let $A = \{1,2,-3\}$ be a set. Let

$R_1 = \{(a, b)) \in A \times A: a \geq b\}$
$= \{(1,1), (2,2), (-3,-3), (1,-3), (2,1), (2,-3)\}$

$R_2 = \{(1,1), (2,2), (-3,-3), (1,2), (2,1)\}$

$R_3 = \{(1,1), (2,2), (-3,-3)\} = I_A$

Are $(A, R_1), (A, R_2)$ and (A, R_3) partially ordered sets?

Example3.75: Show that $(Z, \geq)$ is a partially ordered set

Solution: Let R be a relation such that

$R = \{(a, b) \in Z \times Z : a \geq b\}$

We must show R is reflexive, anti symmetric and transitive

Reflexive: since $a \geq a \implies (a, a) \in \mathrm{R} \implies \mathrm{R}$ is reflexive

anti-symmetric: let $(a, b) \in R \wedge (b, a) \in \mathrm{R}$ T. P. $a = b$

$$\implies a \geq b \implies a = r + b, \ \mathrm{r} \geq 0 \ldots .(\mathbf{1})$$

And $$b \geq a \implies b = s + a, \ \mathrm{s} \geq 0 \ldots .(\mathbf{2})$$

Substitute (1) in (2) $\implies b = s + r + b$

$$\implies s + r = 0 \overset{\mathrm{r,s} \geq 0}{\Longrightarrow} \mathrm{r} = \mathrm{s} = 0$$

Substitute $r = 0$ in (1) $\implies a = b$

R is anti symmetric

Transitive: let $(a, b) \in R \wedge (b, c) \in \mathrm{R} \implies a \geq b \wedge b \geq c$

$$\implies a \geq b \implies a = r + b, \ r \geq 0 \ldots .(\mathbf{1})$$

And $$b \geq c \implies b = s + c, \ \mathrm{s} \geq 0 \ldots .(\mathbf{2})$$

Substitute (2) in (1) $\implies a = (r + s) + c$ and $r + s \geq 0$

$$\implies a \geq c \implies (a, c) \in \mathrm{R} \implies \mathrm{R} \text{ is transitive}$$

Remark3.76: For any nonempty set A, the relation $A \times A$ is not P.O.R

Example3.77: Let $A = Z$ and $R = Z \times Z$. Show that R is not P.O.R

The relation R is an equivalence relation $\implies R$ is symmetric

$\Longrightarrow (a,b) \in$ R and $(b,a) \in R \quad \forall a,b \in R$

But $a \neq b$ (in general)

$\Longrightarrow$ R is not anti symmetric

$\Longrightarrow$ R is not $P.O.R$

Example3.78: (H .W.) Let $A = Z$ and $R_1 = \{(a,b) \in Z \times Z: a|\text{b}\}$

$R_2 = \{(a,b) \in Z \times Z: a \leq b\}$

$R_3 = \{(a,b) \in Z \times Z: a < b\}$

$R_4 = \{(a,b) \in Z \times Z: a > b\}$

Show that **(**$Z, a|$b) is not a partially ordered set

Show that R_2 is P.O.R

Show that R_3 and R_4 are not P.O.R

Example3.79: (H.W.) Let $X = \{1,2,3\}$ and $R = \{(A,B) \in P(X) \times P(X): A \subseteq B\}$

Show that R is a partially ordered relation on $P(X)$

Definition3.80: Let R be a P.O.R on a set A and let $a,b \in A$. Then a,b are called **comparable عنصرين قابلين للمقارنة** with respect to R if $(a,b) \in R$ or $(b,a) \in R$.

Mathematically,

a,b are comparable $\Leftrightarrow \quad (a,b) \in R \vee (b,a) \in R$
a,b are not comparable $\Leftrightarrow \quad (a,b) \notin R \wedge (b,a) \notin R$

<u>Example3.81</u>: Let $A = \{1,2,3\}$, let

$R = \{(1,1), (2,2), (3,3), (1,2), (3,1), (3,2)\}$ be a P.O.R

Find the comparable element in A with respect to R

$1R\,1 \implies 1,1$ are comparable

$2R2 \implies 2,2$ are comparable

$3R3 \implies 3,3$ are comparable

$1R2 \implies 1,2$ are comparable

$3R1 \implies 1,3$ are comparable

$2R3 \implies 2, 3$ are comparable

<u>Example3.82</u>: (H. W.) Let $A = \{3,4,6,8,10\}$ and $R = \{(a, b) \in A \times$ A:a b}

Find the comparable element in A with respect to R

2. <u>Totally Ordered Relation</u> العلاقة المرتبة كليا

A relation R on a set A is called **totally ordered relation (T. O. R)** or **totally ordering** if

1. R is P.O.R

2. a, b are comparable $\forall a, b \in A$

العلاقة على المجموعة A تسمى مرتبة كليا اذا كانت العلاقة مرتبة جزئيا وكل عنصرين في A قابلين للمقارنة بالنسبة للعلاقة R

<u>Example3.83</u>: Let $A = \{1,2,-3\}$ be a set. Let

$R = \{(a, b)) \in A \times A : a \geq b\}$
$= \{(1,1), (2,2), (-3,-3), (1,-3), (2,1), (2,-3)\}$

Is R T.O.R?

Solution: P.O.R: From Example 3.74, R is P.O.R

Comparable: From reflexive relation, each element is comparable with itself

$(1,-3) \in R \Longrightarrow 1,-3$ are comparable

$(2,1) \in R \Longrightarrow 1,2$ are comparable

$(2,-3) \in R \Longrightarrow 2,-3$ are comparable

$\therefore a, b$ are comparable $\quad \forall a, b \in A$

$\therefore R$ is T.O.R

<u>Example3.84</u>: Show that $(Z, \geq)$ is a totally ordered set

Solution: P.O.R: From Example3.75, R is P.O.R

Comparable: T.P. $\quad a \geq b \vee b \geq a \quad \forall a, b \in Z$

Let $a, b \in Z \Longrightarrow \mathrm{a}=\mathrm{b} \vee a>b \vee b>a$

$\Longrightarrow \mathrm{a} \geq \mathrm{b} \vee a \geq b \vee b \geq a$

$\Longrightarrow \mathrm{a} \geq \mathrm{b} \vee b \geq a$

$\Longrightarrow \mathrm{a} R \mathrm{b} \vee b R a$

$\therefore a, b$ are comparable $\forall a, b \in Z$

$\therefore Z$ is T.O.R

<u>Example3.85</u>: (H .W.) Show that $(Z, \leq)$ is a totally ordered set

<u>Example3.86</u>: $(\mathbb{R}, \leq),(\mathbb{R}, \geq),(Q, \geq),(Q, \leq),(N, \leq),(N, \geq)$ are totally ordered sets

<u>Example3.87</u>: Give an example of a P.O.R that is not T.O.R